DIFFERENTIAL EXPRESSIONS WITH MIXED HOMOGENEITY AND SPACES OF SMOOTH FUNCTIONS THEY GENERATE

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ABSTRACT. Let $\{T_1, \dots, T_l\}$ be a collection of differential operators with constant coefficients on the torus $\mathbb{T}^n$. Consider the Banach space X of functions f on the torus for which all functions $T_j f$, $j = 1, \dots, l$, are continuous. Extending the previous work of the first two authors, we analyse the embeddability of X into some space $C(K)$ as a complemented subspace. We prove the following. Fix some pattern of mixed homogeneity and extract the senior homogeneous parts (relative to the pattern chosen) $\{\tau_1, \dots, \tau_l\}$ from the initial operators $\{T_1, \dots, T_l\}$. Let N be the dimension of the linear span of $\{\tau_1, \dots, \tau_l\}$. If $N \geqslant 2$, then X is not isomorphic to a complemented subspace of $C(K)$ for any compact space K.

The main ingredient of the proof of this fact is a new Sobolev-type embedding theorem.

0. Introduction

The space $C^{(k)}(\mathbb{T})$ of k times continuously differentiable functions on the unit circle $\mathbb{T}$ of the complex plane is isomorphic to $C(\mathbb{T})$ (modulo constants, the isomorphism is given by k-fold differentiation). However, it has long been known that, already for the 2-dimensional torus $\mathbb{T}^2$, the situation is different. The understanding of this phenomenon has been increasing gradually, starting with [G] and [H], and then through the work done in [K1], [K2], [KwP], [Si], [PS], [KSi], [KM1], [KM2], and [M].

We begin our discussion directly with the general framework considered in [KM1]. Let $T = (T_1, \dots, T_l)$ be a collection of differential operators with constant coefficients on the torus $\mathbb{T}^n$. This means that each T_j is a linear combination of differential monomials $D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Here $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is a multiindex composed of nonnegative integers. Next, by $\partial_j f$ we mean the operator $g \mapsto \frac{\partial}{\partial t} g(e^{2\pi i t})$, $t \in \mathbb{R}$, applied to f with respect to the jth variable.[1] The quantity $|\alpha| = \alpha_1 + \dots + \alpha_n$ is called the *order* of the differential monomial. The order of a differential operator is the largest order of a differential monomial involved in the operator.

The above collection T gives rise to the following seminorm on the set of trigonometric polynomials in n variables:

$$\|f\|_T = \max_{1 \leqslant j \leqslant l} \|T_j f\|_{C(\mathbb{T}^n)}.$$

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[1] In general, we use the mapping $t \mapsto e^{2\pi i t}$ to parametrise the unit circle; so, the trigonometric system is $\{e^{2\pi i k t}\}_{k \in \mathbb{Z}}$, $t \in [0, 1)$, and the (multiple) Fourier series are handled accordingly.

The Banach space determined by this seminorm (i.e., the result of factorization over the null-space and completion) is denoted by $C^T(\mathbb{T}^n)$ and is called the space of smooth function generated by the collection T.

When T consists of all differential monomials of order at most s, we obtain the classical space $C^{(s)}(\mathbb{T}^n)$ of s times continuously differentiable functions on the n-dimensional torus. It is known that if $n \geqslant 2$ and $s \geqslant 1$, then the bidual of this space does not embed in a Banach lattice as a complemented subspace, see [K2], [KwP]. In particular, this bidual (*a fortiori*, the space itself) is not isomorphic to a complemented subspace of $C(K)$ for any compact space K.

If T is an *arbitrary* finite collection of differential monomials, we deal with the classical *anisotropic* spaces of smooth functions. The isomorphism problem for them was treated in [PS], [Si] and [KSi]. Not giving a precise statement, we signalize that the following dichotomy occurs: if the collection T contains a senior differential monomial (i.e., the monomial whose multiindex dominates coordinatewise all other multiindices involved), then, up to some not quite essential subtleties (see [KSi]), $C^T(\mathbb{T}^n)$ is isomorphic to $C(\mathbb{T}^n)$; otherwise, again, the bidual of $C^T(\mathbb{T}^n)$ is not embeddable complementedly in a $C(K)$-space (more generally, in a Banach lattice).

The importance of the absence of a "senior" operator for nonisomorphism was further emphasized by the results of [KM1], [KM2]. In those papers, the case of a collection T consisting not necessarily of differential monomials was treated for the first time.

The main result of [KM1], [KM2] says the following. Suppose all operators in the collection T are of order not exceeding $s > 0$. In every operator T_j of the collection, we drop its *junior part*, i.e., all differential monomials of order strictly smaller than s. The remaining *senior part* τ_j is a homogeneous differential operator of order precisely s. *If there are two linearly independent operators among the τ_j, $j = 1, \ldots, l$, then the bidual of $C^T(\mathbb{T}^n)$ is not isomorphic to a complemented subspace of a $C(K)$-space.*

But if all τ_j are multiples of one of them, the situation was still unclear. More precisely, in the case of two-dimensional torus, in [KM1] it was shown that then the space $C^T(\mathbb{T}^2)$ is isomorphic indeed to a $C(K)$-space if the junior parts of all T_j's vanish. (In higher dimension the picture is more complicated.) However, if they do not, nonisomorphism can again occur. We already saw this when we discussed anisotropic spaces of smooth functions: two incomparable maximal[2] monomials involved in the definition of such a space need not be of one and the same order s, though the space itself is definitely not of type $C(K)$ if such two monomials exist.

This suggests that the concept of mixed homogeneity (permeating the theory of anisotropic spaces) may play a role also in the general situation. The main result of this paper says that it is indeed the case. This can be viewed as a joint refinement of the results of [PS], [Si], [KSi] and [KM1], [KM2].

Geometrically, a *mixed homogeneity pattern* in n variables is determined by a hyperplane Λ intersecting the n positive coordinate semiaxes. The equation of such a hyperplane is $\sum_{k=1}^{n} \frac{x_k}{a_k} = 1$, where the a_k are positive numbers. A differential operator S is said to be Λ-homogeneous if all points corresponding to the multiindices of differential monomials involved in S belong to Λ. Consider a finite collection

[2]The terms "incomparable" and "maximal" are related to the coordinatewise partial ordering of multiindices.

$T = \{T_1, \dots, T_l\}$ of differential operators in n variables such that

$$\sum_{k=1}^{n} \frac{\alpha_k}{a_k} \leqslant 1 \tag{0.1}$$

for every differential monomial $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ involved in at least one of T_j. Dropping all terms in T_j for which we have strict inequality in (0.1), we obtain a differential operator τ_j, to be called the Λ-senior part of T_j.

Theorem 0.1. *If for at least one choice of Λ there are at least two linearly independent operators in the collection of Λ-senior parts, then the bidual of $C^T(\mathbb{T}^n)$ is not isomorphic to a complemented subspace of a $C(K)$-space.*

We want to say immediately that this theorem reduces easily to the case of the two-dimensional torus. This requires slight concentration, but similar procedures were described in [KSi], [KM1], and [PS], so we only give a hint to the proof. The space $C^T(\mathbb{T}^n)$ is determined in fact by the linear span of the collection T; so, if this collection gives rise to two linearly independed Λ-senior parts, there is no loss of generality in assuming the existence of two nonequal differential monomials, each occurring in precisely one of these Λ-senior parts. Now, a guideline for the reduction is to draw the 2-plane through the origin and the points corresponding to the multiindices of these two monomials. Consult the papers cited above for more details.

So, in what follows we mainly restrict ourselves to the case of $n = 2$. It should be noted that, in all the above references except for the early papers [G] and [H], nonisomorphism was proved by combining the techniques of absolutely summing operators with a Sobolev-type embedding theorem. Here we do the same, more specifically, we try to imitate the pattern of [KM1], [KM2]. However, the required embedding theorem is not quite standard. To present the statement, we remind the reader the definition of a Sobolev space with nonintegral smoothness:

$$W_2^{\alpha,\beta}(\mathbb{T}^2) = \{f \in C^\infty(\mathbb{T}^2)' \colon \{(1+m^2)^{\frac{\alpha}{2}}(1+n^2)^{\frac{\beta}{2}} \hat{f}(m,n)\} \in l^2(\mathbb{Z}^2)\}. \tag{0.2}$$

We need yet another notion.

Definition 0.2. A distribution f on $\mathbb{T}^2$ is said to be proper if $\hat{f}(s,t) = 0$ whenever $s = 0$ or $t = 0$, $s, t \in \mathbb{Z}$.

In particular, we can talk about a proper measure, or (integrable) function, or trigonometric polynomial. Note also that, on the set of proper functions, the norm of the space (0.2) is equivalent to the norm $\|f\|_W = \Big\|\{|m|^\alpha |n|^\beta \hat{f}(m,n)\}\Big\|_{l^2(\mathbb{Z}^2)}$.

Theorem 0.3. *Suppose that proper distributions $\varphi_1, \dots, \varphi_N$ on the 2-dimensional torus satisfy the system of equations*

$$-\partial_1^k \varphi_1 = \mu_0; \quad \partial_2^l \varphi_j - \partial_1^k \varphi_{j+1} = \mu_j, \quad j = 1, \dots, N-1; \quad \partial_2^l \varphi_N = \mu_N, \tag{0.3}$$

where $\mu_0, \dots, \mu_N$ are measures. Then

$$\sum_{j=1}^{N} \|\varphi_j\|_{W_2^{\alpha,\beta}(\mathbb{T}^2)} \leqslant C \sum_{j=0}^{N} \|\mu_j\| \tag{0.4}$$

whenever nonnegative α and β satisfy

$$\frac{\alpha + \frac{1}{2}}{k} + \frac{\beta + \frac{1}{2}}{l} = 1. \tag{0.5}$$

.

Surely, by the norm of a measure we mean its total variation.

There is a similar statement for the plane, which looks like this.

Theorem 0.4. *Let φ_j, $j = 1, \dots, N$, be compactly supported distributions on the plane $\mathbb{R}^2$. Suppose that they satisfy equations* (0.3)*, where $\mu_0, \dots, \mu_N$ are finite* (*compactly supported*) *measures on the plane. Then an analog of* (0.4) *holds true, namely*

$$\sum_{j=1}^{N} \|\varphi_j\|_{W_2^{\alpha,\beta}(\mathbb{R}^2)} \leqslant C \sum_{j=0}^{N} \|\mu_j\|. \tag{0.6}$$

Here still α and β are nonnegative and satisfy (0.5).

The Sobolev space on the plane in this theorem is defined as follows:

$$W_2^{\alpha,\beta}(\mathbb{R}^2) = \{f \in S'(\mathbb{R}^2) \colon |\xi|^\alpha |\eta|^\beta \hat{f}(\xi,\eta) \in L^2(\mathbb{R}^2)\}.$$

This means that we disregard junior derivatives deliberately.

Taken together, Theorems 0.3 and 0.4 constitute the second main result of the paper. They differ from classical statements by the fact that the condition to be a measure (or an integrable function, which is nearly the same in our setting) is imposed on some linear combinations of derivatives of different functions rather than on certain derivatives themselves of one function. If $k = l = 1$, the two theorems were proved in [KM1] by adapting the well-known and elegant Gagliardo–Nirenberg method that yields a limit order embedding in Sobolev's original situation. To be specific, we mean the inequality

$$\|\varphi\|^2_{L^2(\mathbb{R}^2)} \leqslant \|\partial_1 \varphi\|_{L^1(\mathbb{R}^2)} \|\partial_2 \varphi\|_{L^1(\mathbb{R}^2)},$$

where φ is a compactly supported smooth function; we remind the reader that the key point in the method is to write two formulas representing φ as an "indefinite integral" of its derivative of order 1. However, when $k \neq l$, that method fails totally. At present, passage to Fourier transforms and then forcing our way through fairly nasty improper oscillatory integrals seems only be available in the general case. It should however be noted that, ideologically, the claims of Theorems 0.3 and 0.4 can be viewed as a blend of the embedding result in [KM1] and the well-known embedding theorems for anisotropic Sobolev spaces, see [S], [PS].

It is natural to ask what happens if the assumption of Theorem 0.1 is violated, that is, the Λ-senior parts of the differential operators are proportional to some of them for an arbitrary choice of the plane Λ. Here, for simplicity, we again restrict our considerations to the case of $n = 2$. We shall see that, under a certain "ellipticity" condition, the space $C^T(\mathbb{T}^2)$ is isomorphic in this situation to $C(\mathbb{T}^2)$.[3] However, without this ellipticity condition nonisomorphism to a complemented subspace of a $C(K)$-space may occur again, at least for two different reasons. First, a change of variables may sometimes restore the applicability of Theorem 0.1. Second, there is an effect of somewhat different nature, related to the Cohen theorem on idempotents and also leading to nonisomorphism. This effect was first observed in [M] in the case of dimension 3.

[3]We recall that, by the Milyutin theorem, all spaces $C(K)$ for K compact metric and uncountable are mutually isomorphic. So we can always talk about some fixed one, say, $C(\mathbb{T})$ in similar situations

However, our analysis will still be not quite complete. The problems remaining are rather of arithmetic nature. In particular, consider the collection of two operators $T = \{\mathrm{id}, \partial_1 + \sqrt{2}\partial_2\}$. *We do not know whether the space $C^T(\mathbb{T}^2)$ embeds complementedly in a $C(K)$-space.*

The paper is organized as follows. In §1, we explain in detail what we know if the assumption of Theorem 0.1 fails. In §2, we prove Theorem 0.1 on the basis of Theorem 0.3. The arguments will be somewhat similar to but more complicated than those in [KM1], [KM2]. Finally, in §3 we prove Theorem 0.3. For that, we shall first establish Theorem 0.4 (which will be space-consuming) and then reduce Theorem 0.3 to it (which will not be immediate either). The proofs in §3 are mainly due to the third author.

Finally, we address the reader to the monograph [W] for the most part of the Banach space theory stuff mentioned in what follows (the Milyutin theorem, p-absolutely summing operators, the Grothendieck theorem, etc.).

1. Beyond Theorem 0.1

1.1. **Statements.** As has already been mentioned, the "nonisomorphism" Theorem 0.1 reduces to dimension 2. In this section we aim at opposite results, but we still restrict ourselves to dimension 2 for simplicity. As was observed already in [KM2] and [M], new subtleties may emerge in higher dimensions.

For convenience, we restate Theorem 0.1 for $n = 2$. In this case the hyperplane Λ is a straight line. It is clear that, to verify the assumption of Theorem 0.1, we need not search through *all possible* straight lines. Specifically, consider the collection $\mathcal{M} = \{M_1, \dots, M_K\}$ of all points on the plane whose coordinates are biindices of the differential monomials involved in operators that determine the space in question; then only those lines Λ are interesting that pass through at least two points of this collection and, surely, intersect the two positive coordinate semiaxes and have the property that no point among the M_s, $s = 1, \dots, K$, lies above Λ. Such a line is said to be *admissible*. Now, the desired statement looks like this.

Theorem 1.1. *Suppose that, for at least one admissible line Λ, among the Λ-senior parts of the operators in the collection $T = (T_1, \dots, T_l)$ there are at least two linearly independent. Then the bidual of $C^T(\mathbb{T}^2)$ is not isomorphic to a complemented subspace of a $C(K)$-space.*

In this section we are interested in the situation when the assumption of the above theorem is violated. Let us understand what this means. There may be either no or finitely many admissible lines. In the latter case, a concave broken line[4] $\mathcal{Z}$ arises from them, with nodes $Z_t = (x_t, y_t), t = 1, \dots, N$. We may and do assume that the x_t decrease and y_t increase monotonically with t. This means that the indices increase if we move along $\mathcal{Z}$ counterclockwise. The admissible line Λ passing through Z_t and Z_{t+1} will be marked with the index t. Then the Λ_t-senior part of each T_j is the linear combination (with the same coefficients as in T_j) of the differential monomials involved in T_j and such that the points corresponding to their biindices lie on the segment $[Z_t, Z_{t+1}]$. Note that, for every t, the monomial $\partial_1^{x_t}\partial_2^{y_t}$ corresponding to Z_t occurs obligatory in some senior part.

It is convenient to extend the broken line $\mathcal{Z}$ by adding to it two points $Z_0 = (x_1, 0)$ and $Z_{N+1} = (0, y_N)$ and the corresponding segments. Then the initial

[4]We mean that it is the graph of a concave function.

broken line will be referred to as the *core* of the extended one. Each of the two additional segments may degenerate to a point; otherwise the first of them is part of a vertical line and the second is part of a horizontal line; naturally, these two lines *are not regarded as admissible.*

If there are no admissible lines at all, the configuration described above turns into a broken line with three nodes M_0, M_1, M_2; its core is reduced to the point M_1, and the other two points are the projections of M_1 to the coordinate axes. Note that in this case the differential monomial corresponding to M_1 must occur with a nonzero coefficient in at least one T_j.

We define the *principal part* of T_j to be the linear combination, with the same coefficients as in T_j, of the differential monomials corresponding to the points that lie in the core of $\mathcal{Z}$. We omit the easy proof of the following statement.

Lemma 1.2. *If the assumption of Theorem* 1.1 *is violated, then all principal parts of the operators T_j are multiples of one.*

It has already been mentioned that the space $C^T(\mathbb{T}^2)$ is in fact determined by the linear span of the collection T. Now, by the lemma, if the assumption of Theorem 1.1 fails, we can replace the T_j by their linear combinations (without changing the space) in such a way that the only operator with nontrivial principal part is T_1. If admissible lines do exist, we use the fact that the monomial corresponding to an arbitrary node Z_k must occur in some senior part to conclude that then the Λ_t-senior parts S_t of T_1 are nonzero for every t.

Next, the characteristic polynomial P_S of a differential operator $S = \sum_k a_k \partial_1^{\alpha_k} \partial_2^{\beta_k}$ on the two-dimensional torus is defined as follows:

$$P_S(x, y) = \sum_k a_k (2\pi i x)^{\alpha_k} (2\pi i y)^{\beta_k}.$$

Now, if admissible lines exist, we want to impose some *ellipticity condition.*

Definition 1.3. We say that S satisfies the ellipticity condition provided the characteristic polynomials of all senior parts S_t do not vanish on $\mathbb{R}^2$ except, possibly, for the coordinate axes.

Theorem 1.4. *Suppose that the assumption of Theorem* 1.1 *is violated, at least one admissible line exists and, after the modification of the collection T described above, the ellipticity condition is fulfilled. Then $C^T(\mathbb{T}^2)$ is isomorphic to $C(\mathbb{T}^2)$.*

Recall that, by the Milyutin theorem, here we might replace $C(\mathbb{T}^2)$ by $C(K)$ for any uncountable compact metric space K.

Remark 1.5. The arguments in the proof of Theorem 1.4 (to be presented later) can be adjusted to some situations where the ellipticity condition fails. In particular, if there are no admissible lines, the space in question is always isomorphic to a $C(K)$-space.

1.2. **Still beyond.** The proof of Theorem 1.4 will also be postponed slightly to discuss what can happen if neither it, nor Theorem 1.1 is applicable. We start with an example. Consider the space determined by two operators $T_1 = \partial_1^2 + 2\partial_1\partial_2 + \partial_2^2$ and $T_2 = a\partial_1 + b\partial_2$, where a and b are some complex numbers. We recall that ∂_1 and ∂_2 applied to a function f on the torus are the differentiations of $f(e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$ with respect to θ_1 and θ_2.

We see that, directly, neither of the two statements mentioned above can tell us anything about this space. (Clearly, the first operator is equal to $(\partial_1 + \partial_2)^2$ and is not elliptic in the above sense.) But after the change of variables $\theta_1 = t_1 + t_2$, $\theta_2 = t_2$, the operators T_1 and T_2 turn into $(\partial/\partial t_2)^2$ and $(a-b)\partial/\partial t_1 + b\partial/\partial t_2$, respectively. Theorem 1.1 is applicable to this new collection if $a \neq b$ (consider the straight line passing through the points $(1,0)$ and $(0,2)$); thus, the bidual of the space in question does not embed complementedly in a $C(K)$-space. If $a = b$, Remark 1.5 is applicable (alternatively, the reader may consult [KSi] in this case), and we have isomorphism to $C(K)$.

We can formalize the said above as follows. Suppose there is precisely one admissible line Λ, and suppose it is parallel to the bisector of the second and fourth quadrants. This means that we deal with the usual rather than a mixed homogeneity pattern. As before, suppose that only one operator in the collection (specifically, T_1) has nontrivial Λ-senior part

$$L = \sum_{k=0}^{\mu} a_k \partial_1^k \partial_2^{\mu-k}. \tag{1.1}$$

All differential monomials occurring in L are of order μ. We assume that all other differential monomials involved in at least one of the T_j are of order at most $\mu - p$ for some natural number p, and that a monomial of order precisely $\mu - p$ occurs in at least one operator other than T_1 (we may assume that this operator is T_2). Let $\tilde{L}$ denote the part of T_2 consisting of the monomials of order precisely $\mu - p$.

We assign the polynomial $P(t) = \sum_{k=0}^{\mu} a_k t^k$ to L (then $(2\pi i x)^\mu P(y/x)$ is the characteristic polynomial of L), and we do similarly with $\tilde{L}$, obtaining a polynomial $\tilde{P}(t)$. Next, suppose $P(t)$ has a real rational root q of multiplicity α, and q is a root of $\tilde{P}(t)$ of multiplicity β (the case where $\beta = 0$ is not forbidden). Finally, suppose that $\alpha > \beta + p$ (in particular, this inequality implies that $\alpha \geqslant 2$).

Now, after the change of variables $\theta_1 = t_1 - q t_2$, $\theta_2 = t_2$ (this rational substitution should be understood properly on the torus, so as to yield an isomorphic space of smooth functions), the operator $\partial_2 - q\partial_1$ turns into $\partial/\partial t_2$. This implies that, in the new variables, the first α summands will vanish in the expression like (1.1) for L and the next one will be nonzero, and the same will happen for $\tilde{L}$ with α replaced by β. Then (again in the new variables) the straight line Λ passing through the points $(\beta, \mu - p - \beta)$ and $(\alpha, \mu - \alpha)$ is admissible and the new operators T_1 and T_2 have linearly independent Λ-senior parts

The above constructions leans upon the fact that a homogeneous polynomial in two variables expands (over $\mathbb{C}$) in a product of linear factors $x - cy$. (Moreover, it is required that some of the numbers "c" be real and rational; we shall return to this later.) In the case of mixed homogeneity, the situation changes. We show what may happen by considering the space of $C^T(\mathbb{T}^2)$ with a collection T consisting of only one operator S. The following proposition is nearly obvious. In fact, it is valid in arbitrary dimension and was mentioned already in [KM1].

Proposition 1.6. *Let E be the set of all roots of the characteristic polynomial P_S that belong to $\mathbb{Z}^2$. Then the space $C^{\{S\}}(\mathbb{T}^2)$ is isomorphic to the subspace $C_E(\mathbb{T}^2)$ of $C(\mathbb{T}^2)$ that consists of all functions whose Fourier coefficients vanish on E.*

Now, by a commonplace in the Banach space theory (see [M] for more explanations), if the bidual of $C_E(\mathbb{T}^2)$ embeds complementedly in a $C(K)$-space, then the

Fourier multiplier corresponding to the set E is bounded on $C(\mathbb{T}^2)$. By the celebrated Cohen idempotent theorem (see, e.g., the monograph [GM]), this happens if and only if E belongs to the *coset ring* of the group $\mathbb{Z}^2$, i.e., to the ring of sets generated by the cosets of all subgroups of this group.

Clearly, there are operators S for which E does not belong to the coset ring of $\mathbb{Z}^2$. The simplest operator of this sort (by the way, it is mixed homogeneous) is probably

$$S_0 = 2\pi i\partial_1 - \partial_2^2. \tag{1.2}$$

The fact that an infinite subset of a parabola cannot lie in the coset ring is easy; see, however, [M] for a formal pattern proving statements like that. We emphasize that such cases of nonisomorfism should be viewed as "accidental" in our context because they are not quite relevant to the differential structure.

However, we attract the reader's attention to the interesting example for $\mathbb{T}^3$ presented in [M]. Namely, this is the operator $\partial_1^2 + \partial_2^2 - \partial_3^2$, which is homogeneous in the usual sense. The set E for it consists of the Pithagorean triples and, surely, does not belong to the coset ring of $\mathbb{Z}^3$.

It should be noted that we do not know anything about the isomorphic type of the space generated by two operators id and S_0, where S_0 is given by (1.2). But a more challenging example is the pair $\{\mathrm{id},\, \partial_1 - \sqrt{2}\partial_2\}$. Taken alone, the second operator of this pair generates a space isomorphic to $C(\mathbb{T}^2)$ by Proposition 1.6. However, it does not satisfy the ellipticity condition discussed above (that condition required the absence of *real* and not merely *integral* roots for the characteristic polynomial). So, Theorem 1.4 tells nothing about the couple in question, nor does Theorem 1.1. A rational-linear change of variables (as discussed at the beginning of this subsection) also cannot help in this case.

1.3. **Proof of Theorem 1.4.** We address the reader to the beginning of this section to recall certain definitions and some notation. We repeat that the differential polynomials $T_1,\ldots,T_l$ are composed of differential monomials that correspond to integral points lying within the domain in the first quarter under a concave broken line $\mathcal{Z}$. The end-points of $\mathcal{Z}$ are situated on the positive semiaxes, and the nodes of it are denoted by $Z_j = (x_j, y_j)$, $j = 0\ldots, N+1$. The part of the broken line $\mathcal{Z}$ between Z_1 and Z_N is its core. Only T_1 involves differential monomials corresponding to points in the core of $\mathcal{Z}$. They constitute the principal part R of T_1:

$$R = \sum_{j=1}^{N} a_j \partial_1^{x_j}\partial_2^{y_j} + \sum_{j=0}^{N}\sum_{k=1}^{\varkappa_j-1} b_{jk}\partial_1^{x_j+k\frac{x_{j+1}-x_j}{\varkappa_j}}\partial_2^{y_j+k\frac{y_{j+1}-y_j}{\varkappa_j}}. \tag{1.3}$$

Here $\varkappa_j$ is the number of subintervals into which the segment Z_jZ_{j+1} is split by points with integral coordinates (note that, for fixed j, these subintervals are of equal length). The first sum involves precisely the monomials corresponding to the vertices of $\mathcal{Z}$ (surely, the auxiliary vertices Z_0 and Z_{N+1} are not taken into account if they differ from Z_1 and Z_N). By the above discussion, $a_j \neq 0$ for $j = 1,\ldots,N$.

Next, we recall that Λ_j is the (admissible) line passing through Z_j and Z_{j+1}, $j = 1,\ldots,N$. The Λ_j-principal part of T_1 is then given by

$$S_j = a_j\partial_1^{x_j}\partial_2^{y_j} + a_{j+1}\partial_1^{x_{j+1}}\partial_2^{y_{j+1}} + \sum_{k=1}^{\varkappa_j-1} b_{jk}\partial_1^{x_j+k\frac{x_{j+1}-x_j}{\varkappa_j}}\partial_2^{y_j+k\frac{y_{j+1}-y_j}{\varkappa_j}}. \tag{1.4}$$

All differential monomials that correspond to points within the domain bounded by $\mathcal{Z}$ and the segments OZ_0 and $Z_{N+1}O$ (here O denotes the origin) but not lying in the core of $\mathcal{Z}$ are said to be *subordinate* to the operator R. We shall use the same term for arbitrary linear combinations of such monomials. Now, Theorem 1.4 can be restated as follows.

Proposition 1.7. *Suppose that R satisfies the ellipticity condition (see Definition 1.3) and $r_0, r_1, \dots, r_N$ are some operators subordinate to R. Then the space $C^{\{R+r_0,r_1,\dots,r_N\}}(\mathbb{T}^2)$ embeds complementedly in $C(K)$.*

Remark 1.8. In fact, standard arguments show that in our case the space is isomorphic to $C(\mathbb{T})$. We shall not dwell on this.

We shall deduce the proposition from the fact that the contribution of the r_j to the norm is negligible compared to that of R. In fact, this is true on a subspace of finite codimension.

For a natural number M, we denote by $C_M(\mathbb{T}^2)$ the subspace of $C(\mathbb{T}^2)$ consisting of all functions f such that

$$\hat{f}(m,n) = 0 \quad \text{for} \quad |m| \leqslant M, |n| \leqslant M.$$

Next, we denote by $C_{M,0}$ the space of proper continuous functions satisfying the same condition.[5]

Clearly, $C_M(\mathbb{T}^2)$ has finite codimension in $C(\mathbb{T}^2)$ and $C_{M,0}$ has finite codimension in the space of proper continuous functions. Similarly, we can define the space $C_{M,0}^{\{T_1,T_2,\dots,T_n\}}(\mathbb{T}^2)$, again by imposing the above conditions on Fourier coefficients. This subspace is easily seen to be complemented in $C_M^{\{T_1,T_2,\dots,T_n\}}(\mathbb{T}^2)$. As a complement, we can take the sum of a finite-dimensional space and the space X of functions depending on only one of two variables. It is easy to realize that X is isomorphic to $C(\mathbb{T})$. Thus, it suffices to show that $C_{M,0}^{\{T_1,T_2,\dots,T_n\}}(\mathbb{T}^2)$ embeds complementedly in a $C(K)$-space.

Now, we state the required quantitative result.

Proposition 1.9. *Suppose R obeys the ellipticity condition and r is subordinate to R. Then for every $\varepsilon > 0$ there exists a natural number M such that*

$$\|rf\|_{C(\mathbb{T}^2)} \leqslant \varepsilon \|Rf\|_{C(\mathbb{T}^2)} \tag{1.5}$$

for all $f \in C_{M,0}$.

We postpone slightly the proof of this statement to deduce Proposition 1.7 from it. It suffices to show that $C_{M,0}^{\{R+r_0,r_1,\dots,r_N\}}(\mathbb{T}^2)$ is isomorphic to $C_{M,0}$ for large M, because the latter space is complemented in $C(\mathbb{T}^2)$. But theorem 1.9 implies that, for large M, the norms of the spaces $C_{M,0}^{\{R+r_0,r_1,\dots,r_N\}}(\mathbb{T}^2)$ and $C_{M,0}^{\{R\}}(\mathbb{T}^2)$ are equivalent on the set of proper trigonometric polynomials in two variables, which is dense in each of these spaces. So, we must establish isomorphism between $C_{M,0}^{\{R\}}(\mathbb{T}^2)$ and $C_{M,0}$. But, plainly, the mapping $f \mapsto Rf$ from the first space to the second is an isometry onto its image. So, it suffices to show that this image is dense in the second space, and for this it suffices to verify that the characteristic polynomial of R does

[5] Recall that a function h is proper if $\hat{h}(s,t) = 0$ if either $s = 0$ or $t = 0$.

not vanish at any point outside the square $[-M, M]^2$, except for those on coordinate axes, provided M is sufficiently large. (Indeed, then all proper trigonometric polynomials with spectrum outside this square lie in the image of R.)

However, this again follows from Proposition 1.9 applied to the operator id (the differential monomial corresponding to the origin).

Now, we start the proof of Proposition 1.9. It is an immediate consequence of the following multiplier lemma.

Lemma 1.10. *Suppose that the monomial $\partial_1^\alpha \partial_2^\beta$ is subordinate to the operator R. Then the Fourier multiplier whose symbol is given by $\mu(m,n) = 0$ if $m = 0$, or $n = 0$, $P_R(m,n) = 0$, and otherwise*

$$\mu(m,n) = \frac{(2\pi i m)^\alpha (2\pi i n)^\beta}{P_R(m,n)} \tag{1.6}$$

is bounded from $L_0^1(\mathbb{T}^2)$ into itself and from $C_0(\mathbb{T}^2)$ into itself. Moreover, the norm of its restriction to $C_{M,0}$ (and to the space of functions in L^1 whose Fourier coefficients vanish outside $[-M, M]^2$) tends to zero as $M \to \infty$.

Remark 1.11. It will be clear form the sequel that P_R does not vanish outside $[-M, M]^2$ if M is sufficiently large.

We remind the reader that P_R is the characteristic polynomial of R (in other words, this is its symbol if we describe the action of R in terms of Fourier transforms). Next, as usual in this paper, the subscript “0” in the notation for a function class means the subspace of proper functions in this class.

This lemma is rather standard modulo the following crucial estimate; by the way, the estimate implies that the denominator in the above formula does not vanish outside the square $[-M, M]^2$ and the coordinate axes.

Lemma 1.12. *If R obeys the ellipticity condition, then for every node $Z_j = (x_j, y_j)$, $j = 1, \ldots, N$, of the broken line $\mathcal{Z}$ we have*

$$|m|^{x_j} |n|^{y_j} \leqslant C |P_R(m,n)| \tag{1.7}$$

for all **real** *m and n with $\max\{|m|, |n|\}$ sufficiently large, where C is independent of m and n.*

(Note that, naturally, we do not claim anything about the auxiliary nodes Z_0 and Z_{N+1}.)

Now we show how to deduce Lemma 1.10 from Lemma 1.12. We split the multiplier into 4 pieces corresponding to the 4 coordinate quarters. It suffices to prove the boundedness of each of these pieces separately. Since they are similar, we consider only the multiplier D_ν, where $\nu(m,n) = \mu(m,n)$ if $m, n > 0$ and $\nu(m,n) = 0$ otherwise. Next, clearly it suffices to estimate D_ν on the set of proper trigonometric polynomials f in two variables. Then the sum in the following formula is in fact finite:

$$D_\nu(f)(x,y) = \sum_{m,n \geqslant 0} \nu(m,n) \hat{f}(m,n) e^{2\pi i (mx + ny)}.$$

Summation by parts consecutively in the first and the second variable yields

$$D_\nu(f)(x,y) = \sum_{m,n \geqslant 0} S_{m,n}(f) \delta_x \delta_y(\nu)(m,n). \tag{1.8}$$

Here $S_{m,n}(f) = \sum_{0\leqslant k\leqslant m} \sum_{0\leqslant j\leqslant n} \hat{f}(k,j) z_1^k z_2^j$ is a partial sum of the Fourier series of f and δ_x and δ_y are the standard difference operators in the first and the second variable. (Note that, in general, additional terms arise under summation by parts. Here they vanish because $\nu(m,n) = 0$ if $m = 0$ or $n = 0$.)

Since the norm of the operator $S_{m,n}$ (not matter whether it is considered on $C(\mathbb{T}^2)$ or on $L_1(\mathbb{T}^2)$) is of order of $\log(m+1)\log(n+1)$ for $m,n \geqslant 0$, we see that it suffices to prove that the series

$$\sum_{m,n\geqslant 0} \log(m+1)\log(n+1)|\delta_x\delta_y(\nu)(m,n)|$$

converges absolutely and its sum tends to 0 as $M \to \infty$. (Recall that $S_{m,n}(f) = 0$ if m and n are smaller than M, so that summation in (1.8) is in fact over the indices satisfying $\max(|m|,|n|) \geqslant M$.)

The reason why this claim is true is that, when dropping the logarithmic factors, we basically obtain an (absolutely) convergent series of inverse powers, so that logarithms cannot make it divergent. Let us enter in the details.

Clearly, $|\delta_x\delta_y(\nu)(m,n)| \leqslant \max_{m\leqslant s\leqslant m+1, 0\leqslant t\leqslant n+1} |\frac{\partial^2}{\partial_s\partial_t}\nu(s,t)|$. Calculating the above partial derivative in accordance with the usual differentiation rules, we obtain the expression $\frac{Q}{P_R^4}$, where Q is some polynomial. Let $s^p t^q$ be any nonzero monomial occurring in Q, and let

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{1.9}$$

be the equation of some admissible line. It is easy to realize that then

$$\frac{p+1}{a} + \frac{q+1}{b} < 4. \tag{1.10}$$

Indeed, the exponents of the monomial $(2\pi is)^\alpha(2\pi it)^\beta$ satisfy $\frac{\alpha}{a} + \frac{\beta}{b} < 1$, the exponents u, v of any monomial occurring in P_R satisfy $\frac{u}{a} + \frac{v}{b} \leqslant 1$, and, altogether, we multiply four such factors ($(2\pi is)^\alpha(2\pi it)^\beta$ is involved obligatory) but differentiate one of them in s and one (which may or may not be the same) in t when forming a monomial in the numerator.

We estimate from above the quantity $|\frac{s^p t^q}{P_R(s,t)}|$ for $s \in [m, m+1]$, $t \in [n, n+1]$. Consider the point $C = \left(\frac{p+1}{4}, \frac{q+1}{4}\right)$, then it lies within the domain bounded by the broken line $\mathcal{Z}$ and two segments of the coordinate axes. We claim that there exist two neighboring nodes $Z_j = (x_j, y_j)$ and $Z_{j+1} = (x_{j+1}, y_{j+1})$ of $\mathcal{Z}$, $j = 1, \dots, N$ with $\frac{p+1}{4} < x_j$ and $\frac{q+1}{4} < y_{j+1}$.

Indeed, this is clear if C lies on $\mathcal{Z}$ (but, surely, not in its core). Otherwise, consider the lines $x = \frac{p+1}{4}$ and $y = \frac{q+1}{4}$. They hit $\mathcal{Z}$ at two points A and B. If the part of $\mathcal{Z}$ between A and B contains a link or is contained within one link of $\mathcal{Z}$, then the endpoints of this link fit. If this part contains precisely one node of $\mathcal{Z}$, this node can be taken for one of the above two vertices, and either the preceding or the next node can be taken for the other. We shall assume that the line (1.9) passes precisely through these two nodes.

Now, applying Lemma 1.12, we find

$$\left|\frac{s^p t^q}{P_R(s,t)}\right| \leqslant C \frac{s^p t^q}{(s^{x_j} t^{y_j} + s^{x_{j+1}} t^{y_{j+1}})^4}. \tag{1.11}$$

Next, let $\gamma > 0$ be the difference between 4 and the left-hand side of (1.10). The quantities on the right in (1.11) should be multiplied by $\log(s+1)\log(t+1)$ and

then summed, and we must show that the sum over $(s,t) \notin [-M,M]^2$ tends to 0 as $M \to \infty$. However, we take the liberty to act as if the logarithmic factors were absent. Let us sum the above quantities over the pairs (s,t) satisfying $s^{x_j}t^{y_j} \leqslant s^{x_{j+1}}t^{y_{j+1}}$ (the case of the opposite inequality is treated similarly); this condition means in fact that $t \geqslant s^{\frac{a}{b}}$. So, the sum in question is dominated termwise by the double series

$$\sum_s \left(\sum_{t \geqslant s^{a/b}} t^{q-4y_{j+1}} \right) s^{p-4x_{j+1}}.$$

By the above discussion, $q - 4y_{j+1} < -1$, whence the inner sum over t is dominated by $s^{\frac{a}{b}(q-4y_{j+1}+1)}$. Thus, we arrive at the series $\sum_s s^\rho$, where

$$\begin{aligned}&\rho = p - 4x_{j+1} + \frac{a}{b}(q - 4y_{j+1} + 1) = \\ &a\left(\frac{p}{a} - 4\frac{x_{j+1}}{a} + \frac{q}{b} - 4\frac{y_{j+1}}{b} + \frac{1}{b}\right) = a\left(\frac{p+1}{a} + \frac{q+1}{b} - 4\left[\frac{x_{j+1}}{a} + \frac{y_{j+1}}{b}\right] - \frac{1}{a}\right) \\ &\qquad = -1 - a\gamma.\end{aligned}$$

Clearly, when we incorporate the omitted logarithmic factors, we still can dominate the resulting series by a convergent series of inverse powers with slightly smaller exponent.

Now, since the series converges, its sum over $(s,t) \notin [-M,M]^2$ tends to 0, and we are done.

It remains to prove Lemma 1.12.

Consider two differential monomials involved in R and corresponding to the nodes Z_i and Z_j in the core of the broken line $\mathcal{Z}$, that is, the monomials $\partial_1^{x_i}\partial_2^{y_i}$ and $\partial_1^{x_j}\partial_2^{y_j}$. We want to compare the magnitudes of their characteristic polynomials. These polynomials (in fact, monomials) are equal in the absolute value on the set where $|2\pi m|^{x_i}|2\pi n|^{y_i} = |2\pi m|^{x_j}|2\pi n|^{y_j}$, which can be rewritten as $|2\pi m|^{c_{ij}}|2\pi n|^{-d_{ij}} = 1$, where $c_{ij} = x_i - x_j$ and $d_{ij} = y_i - y_j$. Next, their moduli are comparable on the set

$$w_{ij} = \{(m,n) \in \mathbb{Z}^2 \setminus [-M,M]^2 \colon \gamma_{i,j} \leqslant |m|^{c_{ij}}|n|^{-d_{ij}} \leqslant \gamma_{i,j}^{-1}\}, \tag{1.12}$$

where $\gamma_{i,j}$ is a constant not exceeding 1 and to be fixed later. Naturally, w_{ij} splits in four pieces (four combinations of pluses and minuses are possible when we lift the modulus signs). Also, $\mathbb{R}^2 \setminus [-M,M]^2$ becomes split into four parts, each containing a ray of a coordinate axis. In the parts containing rays of the x-axis, the monomial with smaller index dominates the other (that is, for $i < j$, the algebraic monomial corresponding to Z_i dominates the one liked with Z_j), and the opposite is true in two other parts.

The sets w_{ij} are symmetric with respect to each coordinate axis. Therefore, it suffices to understand how they are situated relative to each other in the first quarter. We say that a point $P \in \mathbb{R}^2$ is greater than another point Q if their first coordinates are equal, they lie in one and the same quarter, and the second coordinate of P has larger absolute value than the second coordinate of Q.

Proposition 1.13. *For every collection $\gamma_{i,j}$, $\gamma_{i,j} \in (0,1)$, there exists M so large that every point of w_{ij} is greater than some point of w_{kl}, whenever $i \geqslant k, j > l$.*

Proof. It suffices to verify this in the first quarter. But on w_{ij} we have $n \asymp m^{\frac{c_{ij}}{d_{ij}}}$, whereas on w_{kl} we have $n \asymp m^{\frac{c_{kl}}{d_{kl}}}$. It remains to observe that $\frac{c_{kl}}{d_{kl}} < \frac{c_{ij}}{d_{ij}}$ because the broken line $\mathcal{Z}$ is concave. □

Now consider the domains $w_{j,j+1}$, which "separate" the monomials corresponding to Z_j and Z_{j+1}. Whatever the collection $\gamma_{j,j+1}$ is, there exists a large M such that, outside $[-M,M]^2$, the sets $w_{j,j+1}$ are mutually disjoint and follow one after another in the counterclockwise order when j increases. We denote the set between $w_{j-1,j}$ and $w_{j,j+1}$ by Ω_j. In terms of inequalities, the definition of Ω_j looks like this:

$$\begin{aligned}(1.13)\quad \Omega_j = \{(m,n) \notin [-M,M]^2 \colon \gamma_{j-1,j} \geqslant |m|^{c_{j-1,j}}|n|^{-d_{j-1,j}},\\ |m|^{c_{j,j+1}}|n|^{-d_{j,j+1}} \geqslant \gamma_{j,j+1}^{-1}\}.\end{aligned}$$

Now, Ω_j is the set where the monomial corresponding to Z_j dominates the other monomials. Here is the precise statement.

Proposition 1.14. *Let $\partial_1^x\partial_2^y$ be a monomial involved in R but not corresponding to Z_j (it may correspond either to some other node or to a point on some segment of the broken line), and let λ be a positive number. If $\gamma_{j-1,j}$ and $\gamma_{j,j+1}$ are sufficiently small and M is sufficiently large, we have*

$$(1.14)\qquad |m|^{x_j}|n|^{y_j} \geqslant \lambda |m|^x|n|^y \quad \textit{for} \quad (m,n)\in\Omega_j.$$

Proof. We rewrite the inequality to be verified:

$$(x_j - x)\log|m| + (y_j - y)\log|n| \geqslant \log\lambda.$$

The point $(\log|m|, \log|n|)$ lies in the domain

$$\begin{aligned}\{(x,y) \notin [-\log M, \log M]^2\colon\\ \log\gamma_{j-1,j} \geqslant c_{j-1,j}u - d_{j-1,j}v;\quad c_{j,j+1}u - d_{j,j+1}v \geqslant -\log\gamma_{j,j+1}\}.\end{aligned}$$

Consider the case where (x,y) has smaller argument than Z_j, the symmetric case is treated similarly. Since the broken line $\mathcal{Z}$ is concave, we have $\frac{x_j-x}{y_j-y} + \frac{c_{j-1,j}}{d_{j-1,j}} \geqslant 0$. The definition of Ω_j shows that

$$\log|n| \geqslant \frac{-\log\gamma_{j-1,j} + c_{j-1,j}\log|m|}{d_{j-1,j}},$$

that is,

$$\begin{aligned}(1.15)\qquad (x_j - x)\log|m| + (y_j - y)\log|n| \geqslant\\ (x_j - x)\log|m| + (y_j - y)\frac{-\log\gamma_{j-1,j} + c_{j-1,j}\log|m|}{d_{j-1,j}}\\ \geqslant (y_j - y)\frac{-\log\gamma_{j-1,j}}{d_{j-1,j}},\end{aligned}$$

because the coefficient of $\log|m|$ is nonnegative. Taking $\gamma_{j-1,j}$ sufficiently small, we can make this quantity greater than $\log\lambda$. □

This proposition allows us to prove inequality (1.7) on Ω_j if $\gamma_{j,j-1}$ is small. Indeed, recalling formula (1.3) for R, we see that we must prove the inequality

$$
\begin{aligned}
&|2\pi m|^{x_i}|2\pi n|^{y_i} \leqslant \\
&C\left|\sum_{\theta=1}^{N} a_\theta(2\pi i m)^{x_\theta}(2\pi i n)^{y_\theta} + \sum_{\theta=0}^{N}\sum_{k=1}^{\varkappa_\theta-1} b_{\theta k}(2\pi i m)^{x_\theta+k\frac{x_{\theta+1}-x_\theta}{\varkappa_\theta}}(2\pi i n)^{y_\theta+k\frac{y_{\theta+1}-y_\theta}{\varkappa_\theta}}\right|.
\end{aligned}
$$

By (1.14), on Ω_j we can replace the monomial $|m|^{x_i}|n|^{y_i}$ on the left by $|m|^{x_j}|n|^{y_j}$. Now, we choose λ greater than $a_j^{-1}2N\max_{\theta,k}\varkappa_\theta\max(|a_\theta|,|b_{\theta k}|)$, after which we choose the numbers "γ" so as that to ensure (1.14) with this λ for all monomials involved in R. Then

$$
\begin{aligned}
&\left|\sum_{\theta=1}^{N} a_\theta(2\pi i m)^{x_\theta}(2\pi i n)^{y_\theta} + \sum_{\theta=0}^{N}\sum_{k=1}^{\varkappa_\theta-1} b_{\theta k}(2\pi i m)^{x_\theta+k\frac{x_{\theta+1}-x_\theta}{\varkappa_\theta}}(2\pi i n)^{y_\theta+k\frac{y_{\theta+1}-y_\theta}{\varkappa_\theta}}\right| \geqslant \\
&\qquad |a_j(2\pi i m)^{x_j}(2\pi i n)^{y_j}| - \\
&\left|\sum_{\theta=1,\theta\neq j}^{N} a_\theta(2\pi i m)^{x_\theta}(2\pi i n)^{y_\theta} + \sum_{\theta=0}^{N}\sum_{k=1}^{\varkappa_\theta-1} b_{\theta k}(2\pi i m)^{x_\theta+k\frac{x_{\theta+1}-x_\theta}{\varkappa_\theta}}(2\pi i n)^{y_\theta+k\frac{y_{\theta+1}-y_\theta}{\varkappa_\theta}}\right| \\
&\qquad \geqslant \frac{|a_j|}{2}|m|^{x_j}|n|^{y_j}.
\end{aligned}
$$

It remains to ensure (1.7) on the domains $w_{j,j+1}$. The arguments will be similar.

Proposition 1.15. *Let $\partial_1^x\partial_2^y$ be a monomial of R with (x,y) not belonging to the segment Z_jZ_{j+1} ((x,y) may be a node in the core of $\mathcal{Z}$ or may belong to some link of this core), and let $\lambda > 0$. If M is sufficiently large, for the $\varkappa_j$ fixed above we have*

$$
|m|^{x_j+k\frac{x_{j+1}-x_j}{\varkappa_j}}|n|^{y_j+k\frac{y_{j+1}-y_j}{\varkappa_j}} \geqslant \lambda|m|^x|n|^y \quad \textit{for} \quad (m,n)\in w_{j,j+1}. \tag{1.16}
$$

Note that, on $w_{j,j+1}$, all algebraic monomials corresponding to points on the line Z_jZ_{j+1} are comparable.

The proposition is proved much as the preceding one. The only difference is that the coefficient of $\log|m|$ in (1.15) is strictly positive, because this time the point (x,y) lies strictly below the line Z_jZ_{j+1}. So, we need not impose additional assumptions on $\gamma_{j,j+1}$, it suffices to merely take $|m|$ sufficiently large (this is important, because the constant $\gamma_{j,j+1}$ has already been fixed).

Now, we proceed to the verification of inequality (1.7) on $w_{j,j+1}$. First, the monomial $|m|^{x_i}|n|^{y_i}$ under study can be replaced by any monomial of the form $|m|^{x_j+k\frac{x_{j+1}-x_j}{\varkappa_j}}|n|^{y_j+k\frac{y_{j+1}-y_j}{\varkappa_j}}$ by the last proposition and the above observation that such monomials are comparable. Second, we have the estimate

$$\Big|a_j(2\pi im)^{x_j}(2\pi in)^{y_j} + a_{j+1}(2\pi im)^{x_{j+1}}(2\pi in)^{y_{j+1}} + \sum_{k=1}^{\varkappa_j-1} b_{jk}(2\pi im)^{x_j+k\frac{x_{j+1}-x_j}{\varkappa_j}}(2\pi in)^{y_j+k\frac{y_{j+1}-y_j}{\varkappa_j}}\Big| = \left|(2\pi im)^{x_j}(2\pi in)^{y_j}\prod_{\theta=0}^{\varkappa_j+1}(1+\xi_\theta(2\pi im)^{\frac{-x_{j+1}+x_j}{\varkappa_j}}(2\pi in)^{\frac{y_{j+1}-y_j}{\varkappa_j}})\right| \geqslant C\left|\prod_{\theta=0}^{\varkappa_j+1}\operatorname{Im}(i^{\frac{y_{j+1}-y_j-x_{j+1}+x_j}{\varkappa_j}}\xi_\theta)\right||m|^{x_j}|n|^{y_j}.$$

The first identity is merely a factorization of the polynomial, and the constant in the subsequent inequality is nonzero by the ellipticity condition (the numbers $i^{\frac{y_{\theta+1}-y_\theta-x_{\theta+1}+x_\theta}{\varkappa_\theta}}\xi_\theta$ are not pure real) and the fact that the quantity $|m|^{\frac{-x_{j+1}+x_j}{\varkappa_j}}|n|^{\frac{y_{j+1}-y_j}{\varkappa_j}}$ is bounded away from zero on $w_{j,j+1}$. It follows that for the Λ_j-senior part S_j of R (recall that Λ_j includes the link Z_jZ_{j+1}) and for every λ we can choose M so large that, on $w_{j,j+1}$, we shall have

$$|P_R(m,n)| \geqslant \frac{1}{2}\Big|S_j(m,n)\Big| \geqslant C|m|^{x_j}|n|^{y_j}.$$

This completes the proof.

1.4. **Other isomorphism cases.** The idea of domination allows us to establish isomorphism even in some cases when the ellipticity condition is violated. The simplest one is the case mentioned in Remark 1.5.

Proposition 1.16. *If there are no admissible lines, the space $C^T(\mathbb{T}^2)$ is complemented in a $C(K)$-space.*

In fact, the space in question is isomorphic to $C(\mathbb{T})$, but we refrain from discussing this.

Proof. As previously, it suffices to prove that the subspace $C_0^T(\mathbb{T}^2)$ of admissible functions is complemented in a $C(K)$-space. We may and do assume that $T_1 = R+r$ where R is a differential monomial and both r and all operators $T_2,\dots,T_j$ are subordinate to R. It suffices to show that, for some M, the norm of $C_0^T(\mathbb{T}^2)$ is equivalent to that of $C_0^{\{R\}}(\mathbb{T}^2)$ on the set of proper functions that have no spectrum in the square $[-M,M]\times[-M,M]$. But it is easy to see that if a differential monomial ρ is subordinate to R, then $\|\rho f\|_{C(K)} \leqslant \varepsilon\|Rf\|_{C(K)}$ for every such f if M is sufficiently large. □

We present yet another related example of isomorphism. Let $e_1, e_2, \dots, e_n$ be pairwise nonproportional rational vectors on the plane. In what follows, we need a different notion of a proper function. Specifically, a function is said to be *quite proper* if its Fourier coefficients vanish at all points of $\mathbb{Z}^2$ that lie on the straight lines generated by the above vectors. In [KM1] it was explained that passage to quite proper functions does not change the isomorphic type of the spaces we treat here. For any quite proper function $f \in C^T(\mathbb{T}^2)$, we can find a function $g \in C^T(\mathbb{T}^2)$ such that $\partial_{e_k}g = f$ (∂_{e_k} denotes a directional derivative) and $\|g\|_\infty \leqslant C\|f\|_\infty$. (By $\|\cdot\|_\infty$ we mean the supremum norm.) This operation will be called formal integration in

the direction e_k, but in fact it is done on the level of Fourier coefficients. Staying within quite proper functions, we avoid division of nonzero Fourier coefficients by zero under this operation.

Now, suppose that the family T contains the operator $T_1 = \partial_{e_1}\partial_{e_2}\ldots\partial_{e_n}$ and all other operators in the family are of order strictly smaller than n. *Then $C^{(T)}(\mathbb{T}^2)$ is isomorphic to $C(\mathbb{T}^2)$. On the quite proper functions, an isomorphism is given by the mapping $f \mapsto T_1 f$.*

For the proof, it suffices to show that $\|T_j f\|_\infty \leqslant C\|T_1 f\|_\infty$ for all $T_j \in T$ if f is quite proper, with C independent of f. To do this, we need a simple algebraic lemma.

Lemma 1.17. *Let $P(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$ be a polynomial of degree n that has n pairwise different roots. Then its monic divisors of degree $n-1$ form a basis in the linear space of polynomials of degree at most $n-1$.*

Proof. Let $P_k(x)$ denote the polynomial $\frac{P(x)}{(x-x_k)}$. When k runs from 1 to n, we obtain all divisors of $P(x)$ of degree $n-1$. It suffices to prove that they are linearly independent. Suppose the contrary, let $\lambda_1 P_1(x) + \lambda_2 P_2(x) + \ldots + \lambda_n P_n(x) = 0$ for some coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$. Putting $x = x_k$, we obtain $\lambda_k = 0$ because all roots are different. □

Now, we prove the above claim. We denote by $T_j^{(n-1)}$ the sum of all terms of order $n-1$ that occur in $T_j \in T$ (some of these "senior parts" of the T_j may be equal to zero). Next, denote by T_{1k} the operator $\frac{T_1}{\partial_{e_k}}$ (i.e., the product of all directional derivations ∂_{e_i} for $i \neq k$). We observe that, on the quite proper functions f, we have $\|T_{1k} f\|_\infty \leqslant C\|T_1 f\|_\infty$ by directional formal integration. But by the lemma, each operator $T_j^{(n-1)}$ is a linear combination of the T_{1k} whence we obtain a similar estimate for all $T_j^{(n-1)}$.

Now we extract the parts of order $n-2$ from all T_j, and estimate them in a similar way by the same method. Then we do the same in the order $n-3$, etc.

2. Nonisomorphism

In this section, we deduce Theorem 1.1 from Theorem 0.3. This will be an improvement of similar arguments in [KM1] and [KM2]. We begin with a solvability condition for system (0.3). We observe that, since all φ_j in this system are assumed to be proper, all measures μ_j must also be proper.

Lemma 2.1. *Suppose $\mu_0, \ldots, \mu_N$ are proper distributions on the torus and $k, l \in \mathbb{N}$. Then system* (0.3) *admits a solution in proper distributions $\varphi_0, \ldots, \varphi_N$ if and only if*

$$\sum_{j=0}^{N} \partial_1^{jk}\partial_2^{(N-j)l}\mu_j = 0. \tag{2.1}$$

Moreover, if (2.1) *is fulfilled, this solution is unique.*

Proof. The implication (0.3)⇒(2.1) is easy. The converse implication is proved by induction on N. The base ($N = 1$) looks like this: if two proper distributions u and v on $\mathbb{T}^2$ satisfy

$$\partial_2^l u + \partial_1^k v = 0, \tag{2.2}$$

then there is a proper distribution w with $u = -\partial_1^k w$ and $v = \partial_2^l w$. This can be verified, e.g., by considering Fourier coefficients. To pass from $N-1$ to N, we rewrite (2.1) in the form (2.2) with $u = \partial_2^{(N-1)l}\mu_0$ (then it is clear what v is), find w as above, and invoke the inductive hypothesis.

Uniqueness is transparent from the above arguments. Alternatively (which is in fact the same), it is easy to see directly from system (0.3) that the Fourier coefficients of the φ_j corresponding to all couples of nonzero integers are determined by those of the μ_j. □

2.1. **Several reductions.** Now, suppose we are under the assumptions of Theorem 1.1. The space $C_0^T(\mathbb{T}^2)$ is complemented in $C^T(\mathbb{T}^2)$, so it suffices to prove that the bidual of $C_0^T(\mathbb{T}^2)$ does not embed complementedly in a $C(K)$-space. Next, let the admissible line Λ mentioned in Theorem 1.1 be given by the equation $\frac{x}{a} + \frac{y}{b} = 1$. We claim that there is no loss of generality in assuming that a and b are natural numbers. Indeed, Λ must contain two points (r_1, r_2) and (ρ_1, ρ_2) with nonnegative integral coordinates, and we may assume that $r_1 > \rho_1$ and $r_2 < \rho_2$. Then the equation of Λ can also be written in the form $\frac{x-\rho_1}{r_1-\rho_1} = \frac{y-\rho_2}{r_2-\rho_2}$, or

$$\frac{x}{r_1-\rho_1} + \frac{y}{\rho_2 - r_2} = \frac{\rho_1}{r_1-\rho_1} + \frac{\rho_2}{\rho_2-r_2}.$$

Next, if $u, v \in \mathbb{N}$, then the spaces

$$C_0^{\{T_1,\dots,T_l\}}(\mathbb{T}^2) \quad \text{and} \quad C_0^{\{T_1\partial_1^u\partial_2^v,\dots,T_l\partial_1^u\partial_2^v\}}(\mathbb{T}^2)$$

are isomorphic. Specifically, the operator $f \mapsto \partial_1^u\partial_2^v f$ is an isomorfism from the second space onto the first. Passage to the new space leads to shifting the line Λ by the vector (u, v). The equation of the shifted line is

$$\frac{x}{r_1-\rho_1} + \frac{y}{\rho_2 - r_2} = \frac{\rho_1+u}{r_1-\rho_1} + \frac{\rho_2+v}{\rho_2-r_2}.$$

The lengths of the segments cut by the new line from the x- and y-axes are equal to

$$\rho_1 + u + (\rho_2+v)\frac{r_1-\rho_1}{\rho_2-r_2} \quad \text{and} \quad (\rho_1+u)\frac{\rho_2-r_2}{r_1-\rho_1} + \rho_2 + v.$$

Clearly, these numbers are integers for some choice of u and v.

So, we assume that $a, b \in \mathbb{N}$. Let N be the greatest common divisor of a and b. We put $m = a/N$, $n = b/N$ (thus, m and n are coprime). All points on Λ whose coordinates are nonnegative integers are of the form $(jm, (N-j)n)$, $j = 0, \dots, N$.

2.2. **Annihilator and the Grothendieck theorem.** Consider the natural embedding $f \mapsto \{T_1 f, \dots, T_l f\}$ of the space $C_0^{\{T_1,\dots,T_l\}}(\mathbb{T}^2)$ into the direct sum of l copies of the space $C(\mathbb{T}^2)$, and let $\mathcal{X}$ be the image of $C_0^{\{T_1,\dots,T_l\}}(\mathbb{T}^2)$. By a Banach space theory commonplace, if the bidual of $C_0^{\{T_1,\dots,T_l\}}(\mathbb{T}^2)$ is isomorphic to a complemented subspace of a $C(K)$-space, then the annihilator of $\mathcal{X}$ in $C(\mathbb{T}^2) \oplus \dots \oplus C(\mathbb{T}^2)$ is complemented in the dual space, which consists of l-tuples of measures on $\mathbb{T}^2$. By the Grothendieck theorem, then an arbitrary bounded linear operator from $\mathcal{X}^\perp$ to a Hilbert space would have been 1-absolutely summing, and, *a fortiori*, 2-absolutely summing. We shall show this is not the case. (The definition of a 2-absolutely summing operator will be given later, when it is really used.)

In order to construct an operator with the required property, we need to understand the structure of the annihilator $\mathcal{X}^\perp$. First, we observe that we can do

all constructions within proper measures. Indeed, to any measure ρ on $\mathbb{T}^2$, we can assign its *proper part* whose Fourier coefficients are equal to $\hat\rho(s,t)$ if $s\neq 0$ and $t\neq 0$, and are zero otherwise. It is easy to realize that this operation is a bounded projection on the space of measures. Moreover, *if a collection of measures annihilates $\mathcal{X}$, then so does the collection of their proper parts.* In what follows, we shall work only with such collections of proper parts. The operations described below will not produce "improper" objects from such collections.

Next, let τ_j be the Λ-senior part of the operator T_j, and let $\sigma_j = T_j - \tau_j$. A collection $(\nu_1,\dots,\nu_l)$ of (proper) measures on $\mathbb{T}^2$ belongs to $\mathcal{X}^\perp$ if

$$0 = T_1^*\nu_1 + \dots + T_l^*\nu_l = \tau_1^*\nu_1 + \dots + \tau_l^*\nu_l + \sigma_1^*\nu_1 + \dots + \sigma_l^*\nu_l. \tag{2.3}$$

We want to modify the collection T of differential operators without changing the space, by using the fact that the space depends only on the linear span of the T_j. By assumption, there are at least two linearly independent operators among $\tau_1^*,\dots,\tau_l^*$. We write

$$\tau_s^* = \sum_{j=0}^{N} a_{sj}\partial_1^{jm}\partial_2^{(N-j)n} \tag{2.4}$$

and make two steps in reshaping the matrix $\{a_{sj}\}$ to a diagonal form. Let j_0 be the smallest index for which $a_{sj_0}\neq 0$ for at least one s. By reindexing, we may assume that $s=1$; replacing T_1 by its multiple, we may assume that $a_{1j_0}=1$. Next, subtracting a multiple of T_1^* from each T_s^* with $s>1$, we can ensure that $s_{sj_0}=0$ for $s>1$. By assumption, after that an index j_1 with $j_1>j_0$ must exist such that $a_{sj_1}\neq 0$ for some $s>1$. There is no loss of generality in assuming that j_1 is the smallest index with this property, $s=2$, and $a_{2j_1}=1$.

Returning to formula (2.3), we assume that all these modifications have already been done. Now, we want to eliminate the junior terms $\sigma_s^*\nu_s$ in this formula. Let α,β be two nonnegative integers satisfying $\frac{\alpha}{a}+\frac{\beta}{b}<1$. There is a standard way to express the restriction of the differential monomial $\partial_1^\alpha\partial_2^\beta$ to the space of proper functions in terms of ∂_1^a and ∂_2^b. Specifically, for $u\neq 0$ and $v\neq 0$ we have

$$\widehat{\partial_1^\alpha\partial_2^\beta f}(u,v) = \frac{(iu)^{\alpha+a}(iv)^\beta}{(iu)^{2a}\pm(iv)^{2b}}\widehat{\partial_1^a f}(u,v) \pm \frac{(iu)^{\alpha}(iv)^{\beta+b}}{(iu)^{2a}\pm(iv)^{2b}}\widehat{\partial_2^b f}(u,v). \tag{2.5}$$

The sign $+$ or $-$ must be one at the same at all occasions; it is determined by the condition $(-1)^a = \pm(-1)^b$, then the denominators do not vanish anywhere except the point $(0,0)$.

Lemma 2.2. *The Fourier multiplies $I_{\alpha\beta}$ and $J_{\alpha\beta}$ with symbols*

$$\frac{(iu)^{\alpha+a}(iv)^\beta}{(iu)^{2a}\pm(iv)^{2b}} \text{ and } \frac{(iu)^{\alpha}(iv)^{\beta+b}}{(iu)^{2a}\pm(iv)^{2b}} \tag{2.6}$$

are bounded on $L^1_0(\mathbb{T}^2)$, consequently, they take proper measures to (proper) measures.

This lemma is a partial case of lemma 1.10. It suffices to take $(iu)^{2a}\pm(iv)^{2b}$ for the role of P_R.

It should be noted that, in fact, the multipliers in question take also $L^1(\mathbb{T}^2)$ to itself (consequently, send measures to measures). This is also easy.

It is convenient to write formula (2.5) in the form

$$\partial_1^\alpha \partial_2^\beta = \partial_1^a I_{\alpha\beta} + \partial_2^b J_{\alpha\beta}. \tag{2.7}$$

We apply (2.7) to all differential monomials occurring in some σ_j^* in (2.3). Then the sum $\sum_j \sigma_j^* \nu_j$ becomes transformed to $\partial_1^a \lambda_1 + \partial_2^b \lambda_2$, where λ_1 and λ_2 are measures obtained from $\nu_1, \dots, \nu_l$ by certain linear operations. After that, we regroup the summands in (2.3) assembling together everything that involves each particular differentiable monomial $\partial_1^{jm} \partial_2^{(N-j)n}$ $(j = 1, \dots, N)$. So, in the resulting expression this differential monomial will be applied to certain measure μ_j, $j = 0, \dots, N$, and the collection of these measures satisfies equation (2.1). Clearly, the μ_j depend linearly on $\nu_1, \dots, \nu_l$ and are all proper.

Now, Lemma 2.1 shows that system (0.3) admits a unique solution $\varphi_0, \dots, \varphi_N$ in proper distributions, which in fact belong to the Sobolev space $W_2^{\frac{k-1}{2}, \frac{l-1}{2}}(\mathbb{T}^2)$ (see (0.2)) by Theorem 0.3. (We have fixed specific values for the parameters in formula (0.5); this is the only particular case of Theorem 0.3 we are going to use.)

So, a bounded linear operator from $\mathcal{X}^\perp$ to the Hilbert space $W_2^{\frac{k-1}{2}, \frac{l-1}{2}}(\mathbb{T}^2)$ has arisen. We summarise how it acts. Given a collection of measures $\mathcal{X}^\perp$, we assign to it the collection $\{\nu_1, \dots, \nu_l\} \in \mathcal{X}^\perp$ of their proper parts; this collection gives rise to measures $\{\mu_0, \dots, \mu_N\}$ satisfying (2.1), from which we obtain functions φ_j belonging to the Sobolev space mentioned above. As was mentioned at the beginning of this subsection, should the bidual of $C_0^T(\mathbb{T}^2)$ be complemented in some $C(K)$-space, this operator would have been 2-absolutely summing.

2.3. **Contradiction.** We explain how to show that the operator constructed above is not 2-absolutely summing. By definition, an operator $S \colon E \to G$ is 2-absolutely summing if it takes weakly 2-summable sequences to 2-summable sequences. A sequence is said to be 2-summable if the squares of the norms of its elements form an absolutely convergent series. A sequence $\{x_j\}_{j \in E}$ is said to be weakly 2-summable if the series $\sum_j |F(x_j)|^2$ converges for an arbitrary bounded linear functional F on E. Clearly, a bounded orthogonal sequence in a Hilbert space is weakly 2-summable. This allows us to exhibit weakly 2-summable sequences in spaces of measures in the following way. Suppose measures $\sigma_k \in M(K)$ are all absolutly continuous with respect to one measure $\sigma \in M(K)$ and their densities form a bounded orthogonal system in $L^2(\sigma)$; then the σ_k form a weakly 2-summable sequence in $M(K)$. Indeed, the mapping $f \mapsto f d\sigma$ is continuous from $L^2(\sigma)$ to $M(K)$, and the property of being a weakly 2-summable sequence survives under the action of a bounded linear operator.

To arrive at the required contradiction, we shall construct a sequence of elements of $\mathcal{X}^\perp$ enumerated by couples (p, q) of natural numbers. It will be of the form

$$\{\nu_1^{(p,q)}, \dots, \nu_l^{(p,q)}\} = \{z_1^p z_2^q d\lambda, c_{pq} z_1^p z_2^q d\lambda, 0, \dots, 0\}. \tag{2.8}$$

Here λ is the normalized Lebesgue measure on the two-dimensional torus. The coefficients c_{pq} will be uniformly bounded, so the above discussion shows that (2.8) is a weakly 2-summable sequence in the space of l-tuples of measures and, consequently, in its subspace $\mathcal{X}^\perp$. For technical reasons (see below) the indices will be subject to the condition

$$\frac{\delta}{2} q^l \leqslant p^k \leqslant \delta q^l \text{ and } p \geqslant C, \tag{2.9}$$

where δ is sufficiently small and C is sufficiently large.

We shall not need know much about the images $\{\varphi_j^{(p,q)}\}_{j=0,\dots,N}$ of these l-tuples of measures under the operator described above. Specifically, let j_0 be the index that arose under modification of the collection T (see formula (2.4) and the explanations after it). It will be ensured then

$$\varphi_{j_0}^{(p,q)} = (ip)^{-k}\gamma_{p,q}z_1^p z_2^q, \text{ where } \inf_{p,q}|\gamma_{(p,q)}| > 0. \tag{2.10}$$

Then the series

$$\sum_{p,q} \|\varphi_{j_0}^{p,q}\|^2_{W_2^{\frac{k-1}{2},\frac{l-1}{2}}(\mathbb{T}^2)}$$

(summation is over (p,q) satisfying (2.9)) diverges. Indeed, clearly, by (2.10), the norms under the summation sign dominate the quantities $p^{k-1}q^{l-1}p^{-2k}$. By condition (2.9), for every fixed $p \geqslant C$ each admissible value of q is roughly $p^{\frac{k}{l}}$, and the number of admissible q is also roughly $p^{\frac{k}{l}}$. Thus, summation of q^{l-1} over admissible q for p fixed yields roughly $p^{\frac{k}{l}}p^{(l-1)\frac{k}{l}} = p^k$, so that the sum of the above quantities over p, q satisfying (2.9) dominates the sum $\sum_{p\geqslant C} p^{-1} = \infty$.

It remains to exhibit elements of the form (2.8) in the annihilator that satisfy the conditions listed above. For this, we must trace what the corresponding measures $\mu_0, \dots, \mu_N$ (see (2.1)) may look like. First, the measures μ_0 and μ_N may involve summands that have arisen from the junior parts of the operators T_j^* in accordance with formula (2.7). These summands are of the form $\xi_{pq}z_1^pz_2^q + \eta_{pq}c_{pq}z_1^pz_2^q$ for μ_0 and $\rho_{pq}z_1^pz_2^q + \varkappa_{pq}c_{pq}z_1^pz_2^q$ (with other coefficients) for μ_N. The numbers ξ_{pq}, η_{pq}, ρ_{pq}, and $\varkappa_{pq}$ can be expressed in terms of the quantities (2.6) with u and v replaced by p and q.

Lemma 2.3. *For some $\varepsilon > 0$, we have*

$$\xi_{pq},\, \eta_{pq},\, \rho_{pq},\, \varkappa_{pq} = O(p^{-\varepsilon}).$$

(We remind the reader that we have imposed the condition $p^k \asymp q^l$.)

Proof. Consider, for example, the first of the quantities (2.6) with p in place of u and q in place of v. Since $\frac{k}{l} = \frac{a}{b}$, we see that, under our assumptions, this quantity has the same order of magnitude as

$$\frac{p^{\alpha+a}p^{\frac{k}{l}\beta}}{p^{2a}+p^{2b\frac{k}{l}}} = \frac{p^{\alpha+a}p^{\frac{a}{b}\beta}}{2p^{2a}} = \frac{1}{2}p^{(-a+\alpha+\frac{a}{b}\beta)} = \frac{1}{2}p^{-a(1-\frac{\alpha}{a}-\frac{\beta}{b})}.$$

It remains to recall that $\frac{\alpha}{a}+\frac{\beta}{b} < 1$ and that the set of the couples (α,β) is finite. $\square$

Now, we want to pay attention to the (proper) solution of system (0.3) if the collection $\{\mu_0, \dots, \mu_n\}$ has arisen from a collection like in (2.8). The calculations depend on the specific position of the indices $j_0 < j_1$ (that arose when we modified the operators T_j) in the interval $[0, N]$. We present the details for two cases; in the other cases, combination of the arguments below is required.

Case 1: $j_0 = 0, j_1 = N$. Then (0.3) acquires the form

$$-\partial_1^k\varphi_1 = z_1^pz_2^q(1+\xi_{pq}+\eta_{pq}c_{pq}); \tag{2.11}$$

$$\partial_2^l\varphi_j - \partial_1^k\varphi_{j+1} = a_{1j}z_1^pz_2^q, \quad j = 1, \dots, N-1; \tag{2.12}$$

$$\partial_2^l\varphi_N = z_1^pz_2^q(a_{1N}+c_{pq}+\rho_{pq}+\varkappa_{pq}c_{pq}). \tag{2.13}$$

Resolving equation (2.11), we find

$$\varphi_1 = -(ip)^{-k}(1 + \xi_{pq} + \eta_{pq}c_{pq})z_1^p z_2^q. \tag{2.14}$$

If $N = 1$ (that is, a and b are coprime), then all equations (2.12) are absent, and (2.13) yields immediately an equation for c_{pq}:

$$-\frac{(iq)^l}{(ip)^k}(1 + \xi_{pq} + \eta_{pq}c_{pq}) = a_{1N} + \rho_{pq} + (1 + \varkappa_{pq})c_{pq}$$

or

$$c_{pq}\left(1 + \varkappa_{pq} + \frac{(iq)^l}{(ip)^k}\eta_{pq}\right) = -a_{1N} - \rho_{pq} - \frac{(iq)^l}{(ip)^k}(1 + \xi_{pq}).$$

Recall that condition (2.9) imposed on the indices involves two parameters to be chosen, namely, δ and C. In the case in question, any choice of $\delta > 0$ will fit. Fixing some δ, by Lemma 2.3 we choose C so large that for all $p \geqslant C$ the coefficient of c_{pq} in the last equation becomes greater than $1/2$. Then the c_{pq} will be uniformly bounded above. Next, increasing C further if necessary and again invoking Lemma 2.3, we can ensure that the coefficient $1 + \xi_{pq} + \eta_{pq}c_{pq}$ in (2.14) be also greater than $1/2$. This guarantees the required conditions if $N = 1$.

But if $N > 1$, we plug the expression (2.14) for φ_1 in the first equation among (2.12) to obtain

$$-\partial_1^k \varphi_2 = a_{11}z_1^p z_2^q + \frac{(iq)^l}{(ip)^k}(1 + \xi_{pq} + \eta_{pq}c_{pq})z_1^p z_2^q,$$

so that

$$\varphi_2 = -\frac{1}{(ip)^k}\left(a_{11} + \frac{(iq)^l}{(ip)^k}(1 + \xi_{pq} + \eta_{pq}c_{pq})\right) z_1^p z_2^q.$$

Continuing in the same manner, we reach the last equation among (2.12), which yields (we put $t = \frac{(iq)^l}{(ip)^k}$):

$$\varphi_N = -\frac{1}{(ip)^k}(Q(t) + t^{N-1}(1 + \xi_{pq} + \eta_{pq}c_{pq}))z_1^p z_2^q,$$

where Q is a polynomial of degree at most $N - 2$ with coefficients depending on the quantities a_{1j}. Then (2.13) implies the following equation for c_{pq}:

$$-tQ(t) - t^N(1 + \xi_{pq} + \eta_{pq}c_{pq}) = a_{1N} + \rho_{pq} + (1 + \varkappa_{pq})c_{pq}$$

or

$$c_{pq}(1 + \varkappa_{pq} + t^N\eta_{pq}) = -tQ(t) - t^N(1 + \xi_{pq}) - a_{1N} - \rho_{pq}.$$

Now, choosing δ small and then C large, we again ensure the uniform boundedness of the c_{pq} and also a uniform lower estimate for the coefficient in (2.14).

Case 2: $j_0 > 0$, $j_1 < N$. Also, for definiteness, we assume that every open interval $(0, j_0)$, (j_0, j_1), and (j_1, N) contains a natural number. Then system (0.3)

looks like this:

$$-\partial_1^k\varphi_1 = z_1^p z_2^q(\xi_{pq} + \eta_{pq}c_{pq}); \tag{2.15}$$

$$\partial_2^l\varphi_j - \partial_1^k\varphi_{j+1} = 0, \quad 0 < j < j_0; \tag{2.16}$$

$$\partial_2^l\varphi_{j_0} - \partial_1^k\varphi_{j_0+1} = z_1^p z_2^q; \tag{2.17}$$

$$\partial_2^l\varphi_j - \partial_1^k\varphi_{j+1} = a_{1j}z_1^p z_2^q, \quad j_0 < j < j_1; \tag{2.18}$$

$$\partial_2^l\varphi_{j_1} - \partial_1^k\varphi_{j_1+1} = (a_{1j_1} + c_{pq})z_1^p z_2^q; \tag{2.19}$$

$$\partial_2^l\varphi_j - \partial_1^k\varphi_{j+1} = (a_{1j} + a_{2j}c_{pq})z_1^p z_2^q, \quad j_1 < j < N; \tag{2.20}$$

$$\partial_2^l\varphi_N = (a_{1N} + a_{2N}c_{pq} + \rho_{pq} + \varkappa_{pq}c_{pq})z_1^p z_2^q. \tag{2.21}$$

Resolving equation (2.21), we obtain

$$\varphi_N = \frac{1}{(iq)^l}(a_{1N} + \rho_{pq} + (a_{2N} + \varkappa_{pq})c_{pq})z_1^p z_2^q.$$

Then the last equation among those labeled by (2.20) takes the form

$$\begin{aligned}\partial_2^l\varphi_{N-1} &= \\ &\left[\frac{(ip)^k}{(iq)^l}(a_{1N} + \rho_{pq} + (a_{2N} + \varkappa_{pq})c_{pq}) + a_{1,N-1} + a_{2,N-1}c_{pq}\right] z_1^p z_2^q \\ = &\left[\frac{(ip)^k}{(iq)^l}(\rho_{pq} + \varkappa_{pq}c_{pq}) + a_{1,N-1} + a_{1N}\frac{(ip)^k}{(iq)^l} + (a_{2,N-1} + \frac{(ip)^k}{(iq)^l}a_{2N})c_{pq}\right] z_1^p z_2^q.\end{aligned}$$

We continue to move “upwards” until we obtain the following equation for φ_{j_1+1} from first equation among the group (2.20) (we again put $t = (iq)^l/(ip)^k$):

$$\partial_2^l\varphi_{j_1+1} = \left[\left(\frac{1}{t}\right)^{N-j_1-1}(\rho_{pq} + \varkappa_{pq}c_{pq}) + A\left(\frac{1}{t}\right) + B\left(\frac{1}{t}\right)c_{pq}\right] z_1^p z_2^q. \tag{2.22}$$

Here A and B are certain polynomials of degree not exceeding N and with coefficients depending on the quantities a_{1j}, a_{2j} only.

Now we find an equation for φ_{j_1}, moving “down” form (2.15). We have

$$\varphi_1 = -\frac{1}{(ip)^k}(\xi_{pq} + \eta_{pq}c_{pq})z_1^p z_2^q.$$

Next, solving the equations in the group (2.16) consecutively, we arrive at

$$\varphi_{j_0} = -\frac{1}{(ip)^k}t^{j_0-1}(\xi_{pq} + \eta_{pq}c_{pq})z_1^p z_2^q,$$

after which (2.17) yields

$$\varphi_{j_0+1} = -\frac{1}{(ip)^k}\left[1 + t^{j_0}(\xi_{pq} + \eta_{pq}c_{pq})\right] z_1^p z_2^q. \tag{2.23}$$

(Recall that, eventually, we must also ensure that the factor in square brackets in (2.23) be bounded away from zero.) The next equation (which is the first in the group (2.18)) then yields

$$\varphi_{j_0+2} = -\frac{1}{(ip)^k}[a_{1,j_0+1} + t + t^{j_0+1}(\xi_{pq} + \eta_{pq}c_{pq})]z_1^p z_2^q.$$

Continuing, we reach the last equation among (2.18) and obtain

$$\varphi_{j_1} = -\frac{1}{(ip)^k}\left[D(t) + t^{j_1-1}(\xi_{pq} + \eta_{pq}c_{pq})\right] z_1^p z_2^q, \tag{2.24}$$

where, again, D is a polynomial of degree not exceeding N and with coefficients depending on the a_{1j}.

Combining (2.22), (2.24), and (2.19), we obtain an equation for c_{pq}:

$$\begin{aligned} &-t\left[D(t) + t^{j_1-1}(\xi_{pq} + \eta_{pq}c_{pq})\right] \\ &\qquad = \left(\frac{1}{t}\right)^{N-j_1}(\rho_{pq} + \varkappa_{pq}c_{pq}) + \frac{1}{t}A\left(\frac{1}{t}\right) + \frac{1}{t}B\left(\frac{1}{t}\right)c_{pq} + a_{1j_1} + c_{pq} \end{aligned}$$

or

$$\begin{aligned} &c_{pq}\left(1 + \frac{1}{t}B\left(\frac{1}{t}\right) + \left(\frac{1}{t}\right)^{N-j_1}\varkappa_{pq} + t^{j_1}\eta_{pq}\right) = \\ &\qquad\qquad - tD(t) - t^{j_1}\xi_{pq} - \frac{1}{t}A\left(\frac{1}{t}\right) - \left(\frac{1}{t}\right)^{N-j_1}\rho_{pq} - a_{1j_1}. \end{aligned}$$

We must ensure that the coefficient of c_{pq} on the left be bounded away from zero uniformly in p and q. Recall that we have imposed the condition $\delta/2 \leqslant |t| \leqslant \delta$, where $\delta > 0$ is still to be chosen. We fix it so big that $|t^{-1}B(t^{-1})| \leqslant \frac{1}{4}$ for all t with $|t| \geqslant \delta$. Than we invoke the restriction $p \geqslant C$ and, using Lemma 2.3, choose C so large that

$$\left(\frac{2}{\delta}\right)^{N-j_1}|\varkappa_{pq}| + \delta^{j_1}|\eta_{pq}| < \frac{1}{4}$$

for all $p \geqslant C$. This ensures a uniform upper bound for the c_{pq}. Increasing C further if necessary, we ensure also that the factor in square brackets in (2.23) be bounded away from zero. So, we are done.

To accomplish our task, it remains to establish the embedding theorem (Theorem 0.3). This will be done in the next section.

3. Embedding theorems

3.1. **On the plane.** As has already been said, we start with proving Theorem 0.4; then we shall use it to verify Theorem 0.3.

3.1.1. *Several observations.* We start with the observation that it suffices to prove Theorems 0.4 and 0.3 only in the case where one of the parameters α, β is zero. Indeed, the general case can be deduced from these two by applying the Cauchy inequality to Fourier images. Next, convolving with an approximate identity, we may replace the measures μ_j with infinitely differentiable compactly supported functions m_j and assume that all φ_j are also compactly supported and infinitely differentiable *a priori*. For definiteness, we assume that $\alpha = 0$, then, by (0.5), $\beta = l - \frac{1}{2} - \frac{l}{2k}$.

Like in [KM1], it suffices to prove the following statement. We recall the standard notation $\mathcal{D}$ for the space of infinitely differentable compactly supported functions on the plane.

Theorem 3.1. *Let $\sigma, \tau \in \mathbb{C}$ be two nonzero complex numbers satisfying $\tau \neq (-1)^{l-k}\overline{\sigma}$. Suppose that functions $f, g, f_1, g_1 \in \mathcal{D}(\mathbb{R}^2)$ satisfy the relations*

$$(\partial_1^k - \tau\partial_2^l)f_1 = f; \quad (\partial_1^k - \sigma\partial_2^l)g_1 = g. \tag{3.1}$$

Then

$$\left|\langle f_1, g_1\rangle_{W_2^{0,l-\frac{1}{2}-\frac{l}{2k}}(\mathbb{R}^2)}\right| \leqslant C_{\tau,\sigma} \|f\|_{L^1(\mathbb{R}^2)} \|g\|_{L^1(\mathbb{R}^2)}. \tag{3.2}$$

By angular brackets, we have denoted the scalar product in $W_2^{0,l-\frac{1}{2}-\frac{l}{2k}}(\mathbb{R}^2)$. It should be noted that

$$\langle f_1, g_1\rangle_{W_2^{0,l-\frac{1}{2}-\frac{l}{2k}}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \hat{f_1}(\xi,\eta)\overline{\hat{g_1}(\xi,\eta)}|\eta|^{2l-1-\frac{l}{k}}\, d\xi d\eta. \tag{3.3}$$

When $k = l = 1$ and τ and σ are real, Theorem 3.1 was proved in [KM1]. In this case, both the statement and the proof are much similar to the classical Gagliardo–Nirenberg inequality $\|f\|_{L^2(\mathbb{R}^2)} \leqslant \|\partial_1 f\|_{L^1(\mathbb{R}^2)} \|\partial_2 f\|_{L^1(\mathbb{R}^2)}$ valid for all smooth compactly supported functions on the plane. In the case where $k \neq l$, the proof will be different and much harder. Now we show how to derive Theorem 0.4 from Theorem 3.1.

Suppose we have identities like (0.3) with functions $m_j \in \mathcal{D}(\mathbb{R}^2)$ in place of the measures μ_j, where the φ_j also belong to $\mathcal{D}(\mathbb{R}^2)$. We must prove inequality (0.6), again with the m_j in place of the μ_j. Taking a complex number s, consider the function $\psi_s = \sum_{j=1}^{N} s^j\varphi_j$. If we multiply the jth equation in (0.3) by s^j and then add all $N+1$ resulting equations, we arrive at the identity $(\partial_1^k - s\partial_2^l)\psi_s = M_s$, where $M_s = \sum_{j=0}^{N} s^j m_j$. Writing another such identity with some number t in place of s, we obtain a system of the form (3.1). The L^1-norms of the functions M_s and M_t do not exceed $\max(1, |s|^N, |t|^N) \sum_{j=0}^{N} \|m_j\|_{L^1}$. So, by Theorem 3.1, we obtain

$$\left|\langle \varphi_s, \varphi_t\rangle_{W_2^{0,l-\frac{1}{2}-\frac{l}{2k}}(\mathbb{R}^2)}\right| \leqslant C_{s,t}\left(\sum_{j=0}^{N} \|m_j\|_{L^1}\right)^2.$$

The numbers s, t must satisfy the condition $s \neq (-1)^{l-k}\overline{t}$.

Next, take $2n$ pairwise distinct numbers $\{s_j\}$ and $\{t_j\}$, $j = 1, \ldots, N$, with $s_i \neq (-1)^{l-k}\overline{t_j}$. We know how to estimate the scalar products $\langle \psi_{s_i}, \psi_{t_j}\rangle_{W_2^{0,l-1}(\mathbb{R}^2)}$, and we must estimate the quantities $a_{ij} = \langle \varphi_i, \varphi_j\rangle_{W_2^{0,l-1}(\mathbb{R}^2)}$ (in fact, only for $i = j$, but it would be improper to impose this restriction at the moment).

Now, expanding, we obtain

$$\langle \psi_{s_i}, \psi_{t_j}\rangle_{W_2^{0,l-1}(\mathbb{R}^2)} = \sum_{(p,q)\in[1,N]^2} (s_i)^p(t_j)^q a_{pq},$$

where $i, j = 1, \ldots, N$. Thus, we have a system of N^2 linear equations for the numbers a_{pq}. The matrix of this system is the tensor product of the matrices $(s_i^p)_{i,p}$ and $(t_j^q)_{j,q}$; its determinant is the product of two Vandermonde determinants to order N. Resolving this system, we immediately arrive at the required estimate.

3.1.2. *The plot.* Until the end of Subsection 3.1, we shall deal with Theorem 3.1. Relations (3.1) allow us to write out an explicit integral formula for the scalar product to be estimated, by using the Fourier transformation and its inverse. Unfortunately, this formula involves complicated improper integrals in two variables. Invoking residue calculus, we reduce them to certain one-dimensional integrals, somewhat similar to the Frullani integral but more complicated. Appropriate estimates for these integrals will finish the proof.

In what follows, we shall present the details under certain fairly strict conditions on σ and τ in Theorem 3.1. However, these conditions distinguish the worst possible case, and for other values of the parameters the arguments are even simpler. Also, we shall assume that k is odd. This will shorten the formulas somewhat; at the end of the proof it will be explained what to do if k is even.

3.1.3. *Reduction to a two-dimensional improper integral.* The functions f_1 and g_1 in (3.2) are infinitely differentiable and compactly supported *a priori*. Thus, their Fourier transforms decay rapidly at infinity, and the integral on the right in (3.3) exists in the usual Lebesgue sense. So, this integral is the limit of integrals over expanding domains exhausting $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} = \lim_{\substack{\varepsilon,\delta\to 0\\ R\to\infty}} \int_{\Omega_{\varepsilon,\delta,R}} .$$

Three parameters in the notation of the domains of integration are a matter of convenience. Specific shape of the domains $\Omega_{\varepsilon,\delta,R}$ will be indicated later. At the moment we tell only that *they are all bounded*. Also, they will be chosen to bypass the singularities that will emerge shortly.

Specifically, singularities arise when we replace the Fourier transforms in the integrand by their expressions in terms of f and g found from the formulas

$$((2\pi i\xi)^k - \tau(2\pi i\eta)^l)\hat{f}_1(\xi,\eta) = \hat{f}(\xi,\eta); \quad ((2\pi i\xi)^k - \sigma(2\pi i\eta)^l)\hat{g}_1(\xi,\eta) = \hat{g}(\xi,\eta),$$

which is the Fourier image of (3.1). So, we shall estimate the quantity

$$\lim_{\substack{\varepsilon,\delta\to 0\\ R\to\infty}} \int_{\Omega_{\varepsilon,\delta,R}} \frac{|\eta|^{2l-1-\frac{l}{k}}\hat{f}(\xi,\eta)\overline{\hat{g}(\xi,\eta)}d\xi d\eta}{((2\pi i\xi)^k - \tau(2\pi i\eta)^l)((-2\pi i\xi)^k - \overline{\sigma}(-2\pi i\eta)^l)}.$$

We assume that every domain $\Omega_{\varepsilon,\delta,R}$ does not intersect some neighborhood (depending on ε, δ, and R) of the set where the denominator in the integrand vanishes. Then (for ε, δ, and R fixed) it is safe to replace $\hat{f}$ and $\hat{g}$ in the last formula by their definitions in terms of f and g, and then change the order of integration:

$$\lim_{\substack{\varepsilon,\delta\to 0\\ R\to\infty}} \iint_{\substack{\operatorname{supp} f\\ \times \operatorname{supp} g}} \int_{\Omega_{\varepsilon,\delta,R}} \frac{|\eta|^{2l-1-\frac{l}{k}} e^{2\pi i((x_1-y_1)\xi+(x_2-y_2)\eta)}}{((2\pi i\xi)^k - \tau(2\pi i\eta)^l)((-2\pi i\xi)^k - \overline{\sigma}(-2\pi i\eta)^l)} d\xi d\eta \; f(x_1,x_2)\overline{g(y_1,y_2)}dy_1dy_2dx_1dx_2.$$

To prove Theorem 3.1, we must show that the modulus of this quantity does not exceed $C\|f\|_1\|g\|_1$. Now, we simply write

$$\left|\lim_{\substack{\varepsilon,\delta\to 0\\ R\to\infty}} \iint\limits_{\substack{\operatorname{supp} f\\ \times \operatorname{supp} g}} H_{\varepsilon,\delta,R}(x,y) f(x)\overline{g(y)}\,dxdy\right| \leqslant$$

$$\varlimsup_{\substack{\varepsilon,\delta\to 0\\ R\to\infty}} \operatorname{ess\,sup}_{\substack{x\in\operatorname{supp} f\\ y\in\operatorname{supp} g}} |H_{\varepsilon,\delta,R}(x,y)|\|f\|_1\|g\|_1,$$

where $x=(x_1,x_2), y=(y_1,y_2)$, and $H_{\varepsilon,\delta,R}(x,y)$ is the following integral:

$$\int\limits_{\Omega_{\varepsilon,\delta,R}} \frac{|\eta|^{2l-1-\frac{l}{k}} e^{2\pi i(a\xi+b\eta)}\,d\xi d\eta}{((2\pi i\xi)^k - \tau(2\pi i\eta)^l)((-2\pi i\xi)^k - \overline{\sigma}(-2\pi i\eta)^l)}. \tag{3.4}$$

Here $a = x_1 - y_1, b = x_2 - y_2$. It will be convenient to introduce yet another auxiliary parameter, namely, assume that the supports of f and g are included in the square centred at the origin, with side length $K/2$, and with sides parallel to coordinate axes. Then $|a| \leqslant K$, $|b| \leqslant K$.

So, we need an asymptotic upper estimate of the quantity (3.4), uniform in a and b.

3.1.4. *Reduction to estimates for one-dimensional integrals.* We are going to represent the integral (3.4) as a sum of one-dimensional improper integrals and certain correction terms; the latter will be uniformly bounded or, in some cases, will even tend to 0 as $\varepsilon,\delta\to 0, R\to\infty$. First, we redenote the constants in the denominator to lighten the notation:

$$((2\pi i\xi)^k - \tau(2\pi i\eta)^l)((-2\pi i\xi)^k - \overline{\sigma}(-2\pi i\eta)^l) = (2\pi i)^{2k}(-1)^k(\xi^k - \tau_1\eta^l)(\xi^k - \sigma_1\eta^l),$$

where $\tau_1 = (2\pi i)^{l-k}\tau$, $\sigma_1 = (-1)^{l-k}(2\pi i)^{l-k}\overline{\sigma}$. The condition $\tau \ne (-1)^{l-k}\overline{\sigma}$ is equivalent to $\sigma_1 \ne \tau_1$. We analyse in detail the case where σ_1 and τ_1 are positive real numbers. The other cases are much similar and even somewhat simpler, the explanations will be given at a due point. It has already been said that we shall assume the oddness of k. If k is even, formulas will differ, but the arguments will remain the same in essence. We shall tell more about that later.

Now, the time is ripe to disclose what $\Omega_{\varepsilon,\delta,R}$ is. We define this domain to be the set of all points $(\xi,\eta)\in\mathbb{R}^2$ such that

$$\eta \in [-R,R]\setminus(-\delta,\delta),$$
$$\xi \in [-R^r, R^r] \quad \text{but}$$
$$\xi \notin \left([(\tau_1\eta^l)^{\frac{1}{k}} - \varepsilon g(\eta), (\tau_1\eta^l)^{\frac{1}{k}} + \varepsilon g(\eta)] \cup [(\sigma_1\eta^l)^{\frac{1}{k}} - \varepsilon g(\eta), (\sigma_1\eta^l)^{\frac{1}{k}} + \varepsilon g(\eta)]\right),$$

where g decays rapidly near zero and near infinity. The number r will be chosen later. For the moment, we assume that $r > \min(1, \frac{l}{k})$. So, we have taken expanding rectangles and cut out certain neighborhoods of the singularities of the integrand in (3.4). It would suffice that g decay like $|\eta|^{l/k}$ at 0 and like $e^{-|\eta|}$ at infinity. Specifically, the following choice of g will fit (we have designed this somewhat

bulky formula in accordance with certain points in the arguments below):

$$g(\eta) = \min\Bigg(\min\big(\sin(\frac{\pi}{2k})\min(\tau_1^{\frac{1}{k}}, \sigma_1^{\frac{1}{k}}), |\sigma_1^{\frac{1}{k}} - \tau_1^{\frac{1}{k}}|\big)|\eta|^{\frac{l}{k}},$$
$$\Bigg(100k\max(K,1)\min\Big(1, \min\big(\sin(\frac{\pi}{2k})\min(\tau_1^{\frac{1}{k}}, \sigma_1^{\frac{1}{k}}), |\sigma_1^{\frac{1}{k}} - \tau_1^{\frac{1}{k}}|\big)|\eta|^{\frac{l}{k}}\Big)^{-2k}\Bigg)^{-1}\Bigg)\times$$
$$\min(e^{\eta}, e^{-\eta}). \tag{3.5}$$

Recall that K measures the size of the supports of f_1 and g_1, in particular, $|a| \leqslant K, |b| \leqslant K$. In fact, *the constant K will only influence the discrepancy estimate rather than the uniform estimate for the limit itself.* So, it is safe to incorporate K in the definition of g. This feature can easily be traced in the arguments below.

We introduce the notation

$$I_{\eta,R,\varepsilon} = \int\limits_{\Gamma_{R,\eta,\varepsilon}} \frac{e^{2\pi i a\xi}d\xi}{(\xi^k - \tau_1\eta^l)(\xi^k - \sigma_1\eta^l)}, \tag{3.6}$$

where

$\Gamma_{R,\eta,\varepsilon} = [-R^r, R^r]\backslash\big([(\tau_1\eta^l)^{\frac{1}{k}} - \varepsilon g(\eta), (\tau_1\eta^l)^{\frac{1}{k}} + \varepsilon g(\eta)] \cup [(\sigma_1\eta^l)^{\frac{1}{k}} - \varepsilon g(\eta), (\sigma_1\eta^l)^{\frac{1}{k}} + \varepsilon g(\eta)]\big)$

is the section of $\Omega_{\varepsilon,\delta,R}$ corresponding to a fixed value of η. Note that two small intervals that we delete are included in $[-R^r, R^r]$ for all admissible η simultaneously if R is large. Indeed, $|(\tau_1\eta^l)^{\frac{1}{k}}| \leqslant R^r - \varepsilon g(\eta), |(\sigma_1\eta^l)^{\frac{1}{k}}| \leqslant R^r - \varepsilon g(\eta)$ for R large, because we have chosen $r > \frac{l}{k}$ and $|\eta| \leqslant R$.

Now, the initial integral takes the form

$$\int\limits_{[-R,R]\backslash(-\delta,\delta)} |\eta|^{2l-1-\frac{l}{k}} e^{2\pi i b\eta} I_{\eta,R,\varepsilon} d\eta, \tag{3.7}$$

and we need to estimate $I_{\eta,R,\varepsilon}$. For definiteness, assume for the moment that $a > 0, \eta > 0$. We perceive ξ as a complex variable. Specifically, the function

$$f_\eta(z) = \frac{e^{2\pi i a z}}{(z^k - \tau_1\eta^l)(z^k - \sigma_1\eta^l)}$$

is meromorphic and decays rapidly in the upper half-plane. We are going to use the residue theorem. For this, we incorporate the integral (3.6) in a contour integral. We have cut out two singularities at the points $(\sigma_1\eta^l)^{\frac{1}{k}}$ and $(\tau_1\eta^l)^{\frac{1}{k}}$; we bypass them clockwise around two small semicircles in the upper half-plane and of radius $\varepsilon g(\eta)$. To make the contour closed, we consider the large semicircle (again in the upper half-plane) centered at zero and of radius R^r. This semicircle will be passed counterclockwise. It will be shown that the integral along the large semicircle tends to zero, whereas the integrals over small semicircles tend to one half of the residue at the corresponding singularity each. Then the integral $I_{\eta,R,\varepsilon}$ turns out to be approximately equal to the sum of the residues inside the contour minus two "half-residues". It is critical, however, to have an appropriate estimate for the discrepancy and to write out the resulting expression explicitly.

It should be noted that, by the choice of g, we may assume that two small half-disks in the definition of the contour do not intersect. Indeed, the distance between their centers is $|(\tau_1\eta^l)^{\frac{1}{k}} - (\sigma_1\eta^l)^{\frac{1}{k}}|$ and the sum of their radii is $2\varepsilon g(\eta)$. So, it suffices to take $\varepsilon \leqslant \frac{1}{2}$.

The function f_η has $k-1$ poles inside the contour, namely, $\frac{k-1}{2}$ poles at the points $(\sigma_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$, $m=1,\ldots,\frac{k-1}{2}$, and the same number of poles at the points $(\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$, $m=1,\ldots,\frac{k-1}{2}$. These points lie indeed inside the contour if R is lager and ε is smaller than some numerical constants. By the residue formula, we obtain

$$I_{\eta,R,\varepsilon} = -\int\limits_{\substack{|z|=R^r\\ \operatorname{Im} z>0}} f_\eta(z)dz + 2\pi i\sum_{m=1}^{\frac{k-1}{2}}\operatorname{Res}_{(\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}} f_\eta + 2\pi i\sum_{m=1}^{\frac{k-1}{2}}\operatorname{Res}_{(\sigma_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}} f_\eta - \int\limits_{\substack{|z-(\tau_1\eta^l)^{\frac{1}{k}}|=\varepsilon g(\eta)\\ \operatorname{Im} z>0}} f_\eta(z)dz - \int\limits_{\substack{|z-(\sigma_1\eta^l)^{\frac{1}{k}}|=\varepsilon g(\eta)\\ \operatorname{Im} z>0}} f_\eta(z)dz. \tag{3.8}$$

We treat all summands separately. First, we consider the integral

$$J_{\eta,R,1} = \int\limits_{\substack{|z|=R^r\\ \operatorname{Im} z>0}} f_\eta(z)dz.$$

Since we need to integrate in η afterwards, we need an accurate estimate for the decay (uniform in a) of this integral. Since $r>\frac{l}{k}$, for R sufficiently large the function f_η admits the following estimate at every point of the semicircle of radius R^r:

$$|f_\eta(z)| = \left|\frac{e^{2\pi iaz}}{(z^k-\tau_1\eta^l)(z^k-\sigma_1\eta^l)}\right| \leqslant \frac{1}{(R^{rk}-\tau_1R^l)(R^{rk}-\sigma_1R^l)},$$

because $|\eta|\leqslant R$. So, the integrand is $O(R^{-2rk})$. The length of the semicircle along which we integrate is πR^r, so finally we obtain

$$|J_{\eta,R,1}| = O(R^{-r(2k-1)}). \tag{3.9}$$

Now, we calculate the residues of f_η. All poles of this function are simple. It is convenient to write

$$f_\eta(z) = \frac{e^{2\pi iaz}}{(z^k-\sigma_1\eta^l)\prod\limits_{m=0}^{k-1}(z-(\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}})}. \tag{3.10}$$

Then the residue $J_{\eta,2,m}$ at $(\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$ is given by the formula

$$J_{\eta,2,m} = \frac{e^{2\pi ie^{2\pi i\frac{m}{k}}a(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)\prod\limits_{n=0,n\neq m}^{k-1}(1-e^{2\pi i\frac{n-m}{k}})} = \frac{e^{2\pi ie^{2\pi i\frac{m}{k}}a(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)}c, \tag{3.11}$$

where c is a scalar coefficient independent of η and m (observe that the product over $n\neq m$ in the denominator is in fact taken over all kth roots of unity except

1). Similarly, denoting by $J_{\eta,3,m}$ the residue of f_η at $(\sigma_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$, we obtain

$$J_{\eta,3,m}=\frac{e^{2\pi i e^{2\pi i\frac{m}{k}}a(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)\prod\limits_{n=0,n\neq m}^{k-1}(1-e^{2\pi i\frac{n-m}{k}})}=\frac{e^{2\pi i e^{2\pi i\frac{m}{k}}a(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)}c.$$

It remains to evaluate the integrals over small semicircles. As has already been mentioned, they are close to one half the residues of f_η at the simple poles $(\tau_1\eta^l)^{\frac{1}{k}}$ and $(\sigma_1\eta^l)^{\frac{1}{k}}$. However, we need to estimate the discrepancy because finally we must integrate in η. The error estimate is given by the following simple lemma.

Lemma 3.2. *If a meromorphic function f has a simple pole at a point x_0 on the real line, then*

$$\left|\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}} f(z)dz+\pi i\operatorname{Res}_{x_0} f\right|\leqslant \pi r\max_{\substack{|z-x_0|\leqslant r\\ \operatorname{Im} z>0}}|h'|,\tag{3.12}$$

where $h(z)=(z-x_0)f(z)$ (the semicircle of integration is directed clockwise, as in (3.8)).

Proof. Since f has simple pole at x_0, it can be written in the form $f(z)=\frac{c}{z-x_0}+f_1(z)$, where f_1 is regular near x_0. Accordingly, we write

$$\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}} f(z)dz=\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}}\frac{cdz}{z-x_0}+\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}} f_1(z)dz.$$

Since $f_1(z)=\frac{h(z)-h(x_0)}{z-x_0}$, the second integral is estimated as follows:

$$\left|\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}} f_1(z)dz\right|\leqslant \pi r\max_{\substack{|z-x_0|\leqslant r\\ \operatorname{Im} z>0}}|f_1|\leqslant \pi r\max_{\substack{|z-x_0|\leqslant r\\ \operatorname{Im} z>0}}|h'|.$$

The first integral can easily be calculated:

$$\int\limits_{\substack{|z-x_0|=r\\ \operatorname{Im} z>0}}\frac{cdz}{z-x_0}=\int\limits_{\substack{|z|=r\\ \operatorname{Im} z>0}}\frac{cdz}{z}=\int\limits_{\frac{1}{2}}^{0}\frac{cdre^{2\pi i\theta}}{re^{2\pi i\theta}}=-\pi ic.$$

□

Now, we denote the integrals over small semicircles by $J_{\eta,\varepsilon,4}$ and $J_{\eta,\varepsilon,5}$, and estimate them. The lemma shows that

$$J_{\eta,\varepsilon,4}=\int\limits_{\substack{|z-(\tau_1\eta^l)^{\frac{1}{k}}|=\varepsilon g(\eta)\\ \operatorname{Im} z>0}} f_\eta(z)dz=-\pi i\operatorname{Res}_{(\tau_1\eta^l)^{\frac{1}{k}}} f_\eta+O\left(\varepsilon g(\eta)\max_{\substack{|z-x_0|\leqslant\varepsilon g(\eta)\\ \operatorname{Im} z>0}}|h'|\right);$$

here

$$h = \frac{e^{2\pi iaz}}{(z^k - \sigma_1\eta^l)\prod_{m=1}^{k-1}(z - (\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}})}$$

by (3.10). Now, h expands in the product of several factors one of which is $e^{2\pi iaz}$ and the other are of the form $(z-p)^{-1}$, where p runs through the poles of f_η different from $(\tau_1\eta^l)^{\frac{1}{k}}$. Differentiating this product, we obtain $2k$ summands, where in each only one factor is replaced by its derivative and the other remain intact. Differentiation of the exponential yields $2\pi iae^{2\pi iaz}$, differentiation of a factor of the second type yields $-(z-p)^{-2}$. The distance between any two poles is at least $\min\big(\sin(\frac{\pi}{2k})\min(\tau_1^{\frac{1}{k}},\sigma_1^{\frac{1}{k}}), |\sigma_1^{\frac{1}{k}} - \tau_1^{\frac{1}{k}}|\big)\eta^{\frac{l}{k}}$. Thus, we arrive at the estimate

$$\max_{\substack{|z-x_0|\leqslant\varepsilon g(\eta)\\ \operatorname{Im} z>0}} |h'| \leqslant$$

$$100k\max(K,1)\min\left(1, \quad \min\big(\sin(\frac{\pi}{2k})\min(\tau_1^{\frac{1}{k}},\sigma_1^{\frac{1}{k}}), \quad |\sigma_1^{\frac{1}{k}} - \eta_1^{\frac{1}{k}}|\big)\eta^{\frac{l}{k}} - \varepsilon g(\eta)\right)^{-2k}.$$

Invoking the definition of $g(\eta)$ by (3.5), we obtain

$$\varepsilon g(\eta)\max_{\substack{|z-x_0|\leqslant\varepsilon g(\eta)\\ \operatorname{Im} z>0}} |h'| \leqslant \varepsilon\min(e^\eta, e^{-\eta}),$$

under the condition $\varepsilon \leqslant \frac{1}{2}$. So,

$$J_{\eta,\varepsilon,4} = -\pi i\operatorname{Res}_{(\tau_1\eta^l)^{\frac{1}{k}}} f_\eta + \varepsilon O(\min(e^\eta, e^{-\eta})).$$

In a similar way, we obtain

$$J_{\eta,\varepsilon,5} = -\pi i\operatorname{Res}_{(\sigma_1\eta^l)^{\frac{1}{k}}} f_\eta + \varepsilon O(\min(e^\eta, e^{-\eta})).$$

So, when $\eta > 0$, formula (3.8) can be rewritten as follows:

$$\begin{aligned}(3.13)\quad I_{\eta,R,\varepsilon} = O(R^{-r(2k-1)}) + 2\pi i\sum_{m=1}^{\frac{k-1}{2}}\frac{e^{2\pi ie^{2\pi i\frac{m}{k}}a(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)}c +\\ 2\pi i\sum_{m=1}^{\frac{k-1}{2}}\frac{e^{2\pi ie^{2\pi i\frac{m}{k}}a(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)}c + \pi i\frac{e^{2\pi ia(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)}c +\\ + \pi i\frac{e^{2\pi ia(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)}c + O(\varepsilon\min(e^\eta, e^{-\eta})).\end{aligned}$$

A similar formula can be written also for $\eta < 0$. If l is even, it will be literally the same, because η occurs in the definition of $I_{\eta,R,\varepsilon}$ only in the power l. But if l is odd, the terms with capital "O" in (3.13) will not change, whereas the other terms will be slightly different, because now the singular points of f_η in the upper half-plane are $(\sigma_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$ and $(\tau_1\eta^l)^{\frac{1}{k}}e^{2\pi i\frac{m}{k}}$, $m = \frac{k+1}{2}, \dots, k-1$. Consequently,

for $\eta < 0$ and l odd, we have

$$I_{\eta,R,\varepsilon} = O(R^{-r(2k-1)}) + 2\pi i \sum_{m=\frac{k+1}{2}}^{k-1} \frac{e^{2\pi i e^{2\pi i \frac{m}{k}} a(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)} c_1 + \tag{3.14}$$

$$2\pi i \sum_{m=\frac{k+1}{2}}^{k-1} \frac{e^{2\pi i e^{2\pi i \frac{m}{k}} a(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)} c_1 + \pi i \frac{e^{2\pi i a(\tau_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\tau_1-\sigma_1)} c_1$$

$$+ \pi i \frac{e^{2\pi i a(\sigma_1\eta^l)^{\frac{1}{k}}}}{\eta^{2l-\frac{l}{k}}(\sigma_1-\tau_1)} c_1 + O(\varepsilon \min(e^{\eta}, e^{-\eta})).$$

These two formulas enable us to rewrite the initial two-dimensional integral as a sum of one-dimensional integrals and small error terms. To do this, we must simply integrate the formulas in η over $[-R,R]\setminus(-\delta,\delta)$, see (3.7). First, we show that the terms with "O" remain small after integration. The contribution of the first summand with "O" is

$$\int\limits_{[-R,R]\setminus(-\delta,\delta)} |\eta|^{2l-1-\frac{l}{k}} e^{2\pi i b\eta} O(R^{-r(2k-1)})\,d\eta,$$

which is $O(R^{2l-\frac{l}{k}-r(2k-1)})$. Choosing $r > \frac{2l-\frac{l}{k}}{2k-1}$, we ensure convergence to zero as $R \to \infty$.

The contribution of the second summand with "O" is

$$\int\limits_{[-R,R]\setminus(-\delta,\delta)} |\eta|^{2l-1-\frac{l}{k}} e^{2\pi i b\eta} O(\varepsilon \min(e^{\eta}, e^{-\eta}))\,d\eta.$$

If we factor out ε, the remaining integrand will be summable over $\mathbb{R}$, so this contribution also tends to 0 as $\varepsilon \to 0$.

Now we look at the fourth and fifth summands. We shall combine them because, separately, they involve nonintegrable singularities at 0 if l is even. But near zero, these summands are very close to be opposite, so that, as we shall see later, these singularities cancel mutually. After substitution in (3.7), these two summands result in the following integral:

$$-\pi i c \int\limits_{[-R,R]\setminus(-\delta,\delta)} |\eta|^{2l-1-\frac{l}{k}} e^{2\pi i b\eta} \frac{(e^{2\pi i a(\sigma_1\eta^l)^{\frac{1}{k}}} - e^{2\pi i a(\tau_1\eta^l)^{\frac{1}{k}}})\,d\eta}{(\sigma_1-\tau_1)\eta^{2l-\frac{l}{k}}}.$$

In the next item we shall explain that, in fact, the integral can be taken over the semiaxis. Now we write the integrals that emerge in a similar way from the second and third summands:

$$-2\pi i c \int\limits_{[-R,R]\setminus(-\delta,\delta)} |\eta|^{2l-1-\frac{l}{k}} e^{2\pi i b\eta} \frac{(e^{2\pi i a e^{2\pi i \frac{m}{k}}(\sigma_1\eta^l)^{\frac{1}{k}}} - e^{2\pi i a e^{2\pi i \frac{m}{k}}(\tau_1\eta^l)^{\frac{1}{k}}})\,d\eta}{(\sigma_1-\tau_1)\eta^{2l-\frac{l}{k}}},$$

where m runs through the values dictated by (3.13) and (3.14) (if l is odd, a more complicated expression arises, because then the value of m changes with the sign of η; however, here again we shall prove that the integrals over each semiaxis are bounded, so this feature is inessential).

Before we proceed further, we explain what happens if σ_1 and τ_1 are not necessarily positive (and real). In a sense, the case we have analysed is the most complicated because, after multiplication of these parameters by complex numbers equal to 1 in modulus, the entire assembly of poles rotates around the origin. As before, only the poles in the upper half-plane will be involved in the residue formula, but, generically, "half-residues" will be absent. In any case, we arrive at one-dimensional integrals of similar nature.

The above arguments pertain to the case of odd k. If k is even, we argue similarly. The difference is that four (rather than two) "half-residues" will emerge, and the number of poles in the upper half-plane will be $k-2$ instead of $k-1$. Again, we shall arrive at similar one-dimensional integrals.

3.1.5. *Estimates for one-dimensional integrals.* We have reduced the problem to estimating one-dimensional integrals of the form

$$\int_{\delta}^{R} e^{2\pi i b\eta}\frac{(e^{2\pi i a_1\varkappa\eta^{\alpha}}-e^{2\pi i a_2\varkappa\eta^{\alpha}})d\eta}{\eta}, \tag{3.15}$$

where $\varkappa$ lies in the upper half-plane and $|\varkappa|=1$, the ratio of a_1 and a_2 is bounded (in our case it is equal to $(\frac{|\sigma_1|}{|\tau_1|})^{\frac{1}{k}}$), and $\alpha>0$ (in our case $\alpha=\frac{l}{k}$). The estimates may depend on $\alpha,\varkappa$ and this ratio, but not on the parameters b,a_1,a_2 themselves.

First, we show that it can be assumed that $\varkappa$ is bounded away from the real axis (to eliminate the special case of the integrals that have arisen from "half-residues"). Observe that, for every α, the integrand admits analytic continuation in a part of an angle with vertex at zero and of opening of roughly $\frac{\pi}{\alpha}$. This part is cut out from the angle by two circles centered at zero and with radii δ and R. It is possible to do this in such a way that the two exponentials $e^{2\pi i a_1\varkappa\eta^{\alpha}}$ and $e^{2\pi i a_2\varkappa\eta^{\alpha}}$ remain bounded by 1 in modulus for all η in this domain. Then the integral (3.15) is equal to minus the integral over a part of the second ray of the angle plus bounded correction terms that arise after integration over two arcs of the circle. An accurate formula looks like this:

$$\int_{\delta}^{R} e^{2\pi i b\eta}\frac{e^{2\pi i a_1\eta^{\alpha}}-e^{2\pi i a_2\eta^{\alpha}}}{\eta}d\eta =$$
$$\int_{\delta}^{R} e^{2\pi i b e^{i\varphi}\eta}\frac{e^{2\pi i a_1(e^{i\varphi}\eta)^{\alpha}}-e^{2\pi i a_2(e^{i\varphi}\eta)^{\alpha}}}{e^{i\varphi}\eta}d(e^{i\varphi}\eta)$$
$$-\int\limits_{\substack{|z|=\delta\\ \operatorname{Arg} z\in[0,\varphi]}} e^{2\pi i bz}\frac{e^{2\pi i a_1z^{\alpha}}-e^{2\pi i a_2z^{\alpha}}}{z}dz+\int\limits_{\substack{|z|=R\\ \operatorname{Arg} z\in[0,\varphi]}} e^{2\pi i bz}\frac{e^{2\pi i a_1z^{\alpha}}-e^{2\pi i a_2z^{\alpha}}}{z}dz.$$

Here φ is the opening of the angle. For instance, we can take $\varphi=\frac{\pi}{2\alpha}$. We see that the first integral is what we need, in it we have $\varkappa=e^{i\varphi\alpha}$, which satisfies our requirements. It remains to show that two remaining integrals are bounded. We

start with the last, which is simpler:

$$\left|\int\limits_{\substack{|z|=R\\ \operatorname{Arg} z\in[0,\varphi]}} e^{2\pi ibz}\frac{e^{2\pi ia_1z^\alpha}-e^{2\pi ia_2z^\alpha}}{z}dz\right| \leqslant \int\limits_{\substack{|z|=R\\ \operatorname{Arg} z\in[0,\varphi]}} 2\left|\frac{dz}{z}\right| \leqslant \frac{2\varphi R}{R} \leqslant \frac{\pi}{\alpha}.$$

The second integral is slightly more involved:

$$\left|\int\limits_{\substack{|z|=\delta\\ \operatorname{Arg} z\in[0,\varphi]}} e^{2\pi ibz}\frac{e^{2\pi ia_1z^\alpha}-e^{2\pi ia_2z^\alpha}}{z}dz\right| \leqslant$$

$$\int\limits_{\substack{|z|=\delta\\ \operatorname{Arg} z\in[0,\varphi]}} |(4\pi(a_1-a_2)z^{\alpha-1}dz| \leqslant 2\pi|a_1-a_2|\alpha\delta^\alpha.$$

Seemingly, this estimate is not quite satisfactory, because it depends on a_1 and a_2. However, it tends to 0 as $\delta\to 0$, and the situation is similar to what we encountered before: only the rate of convergence to the limit rather than the limit itself depends on a_1 and a_2. In fact, this rate of convergence depends only on the parameter K that controls the size of the supports of f, g, f_1, g_1, because the moduli of a_1 and a_2 are a times certain constants depending on σ_1 and τ_1.

Thus it can be assumed indeed that the argument of $\varkappa$ is bounded away from zero. Then the integrand is summable over the entire half-line (it decays exponentially at infinity and is integrable near zero). So, we can forget about δ and R. By a homotety, we can reduce the integral over the half-line to the case where $a_1 = 1, a_2 = c$. In accordance with the discussion at the beginning of this item, c (the ratio of a and b) is controlled in terms of σ_1 and τ_1, i.e., for us it is bounded universally. So, it remains to prove that the following integrals are bounded uniformly in b:

$$\int\limits_0^\infty e^{2\pi ib\eta}\frac{e^{2\pi i\varkappa\eta^\alpha}-e^{2\pi ic\varkappa\eta^\alpha}}{\eta}d\eta,$$

where b is still arbitrary (but its value might have been changed after the change of variables). Substituting $x=\eta^\alpha$, we see that, up to a constant multiple, it is equal to the integral

$$\int\limits_0^\infty e^{2\pi ibx^{\frac{1}{\alpha}}}\frac{(e^{2\pi i\varkappa x}-e^{2\pi ic\varkappa x})dx}{x},$$

which we denote by $\xi(c)$ and treat as a function of c. Clearly, $\xi(1)=0$. We estimate the derivative of ξ:

$$|\xi'(c)| = 2\pi\left|\int\limits_0^\infty e^{2\pi ibx^{\frac{1}{\alpha}}}e^{2\pi ic\varkappa x}dx\right| \leqslant 2\pi\int\limits_0^\infty e^{-c2\pi\operatorname{Im}(\varkappa)x}dx = \frac{1}{\operatorname{Im}(\varkappa)c},$$

whence $|\xi(c)| \leqslant \frac{1}{\operatorname{Im}(\varkappa)}|\log c|$. Recall that, by our assumptions, this quantity is bounded uniformly in a_1, a_2, b if the ratio $\frac{a_1}{a_2}$ is fixed. This finishes the proof of Theorem 3.1 and, with it, of Theorem 0.4.

3.2. **On the torus.** In this subsection we prove Theorem 0.3. It should be noted that, when $k = l = 1$, that theorem was proved in [KM1] *by adjustment of an argument for the plane to the periodic case*. In the present setting, the possibility of such an adjustment is questionable (it is not clear what should replace the residue theorem for functions of discrete argument). So, we shall *reduce* Theorem 0.3 to its planar counterpart Theorem 0.4 rather than adjust the proof.

Since the case of $k = l = 1$ is known, we may assume that $\max(k, l) > 1$ when convenient; this will be required indeed to establish a technical claim (see Lemma 3.5 below).

In the statement of Theorem 0.3, by convolving with approximate identities, we may assume that the measures μ_j and the functions φ_j (see (0.3)) are (proper) trigonometric polynomials. An immediate idea of reduction is to perceive every function on $\mathbb{T}^2$ as a periodic function on the plane. However, Theorem 0.4 applies to compactly supported functions, so that, apparently, we must cut these periodic functions smoothly. Eventually, this will work, but not in the most naive form.

We proceed to the details. First, we need yet another L^1-multiplier lemma.

Lemma 3.3. *Let τ be a complex number such that $i^{l-k}\tau$ is pure imaginary. Then*

$$\|\partial_1^{k-1} f_1\|_{L^1(\mathbb{T}^2)}, \|\partial_2^{l-1} f_1\|_{L^1(\mathbb{T}^2)} \leqslant C_\tau \|f\|_{L^1(\mathbb{T}^2)}, \tag{3.16}$$

where f and f_1 are proper functions (*say, trigonometric polynomials for definiteness*) *related by the first equation in* (3.1), *i.e.,* $(\partial_1^k - \tau\partial_2^l) f_1 = f$.

Proof. We prove the lemma for $\partial_2^{l-1} f_1$, the other case is similar. The Fourier coefficients of f_1 and f are related as follows:

$$\hat{f}_1(m, n) = ((2\pi i m)^k - \tau(2\pi i n)^l)^{-1} \hat{f}(m, n). \tag{3.17}$$

The function $((2\pi i m)^k - \tau(2\pi i n)^l)^{-1}$ is bounded on $\mathbb{Z}^2 \setminus \{(0, 0)\}$ because we have assumed that $i^{k-l}\tau$ is pure imaginary. We see that $\partial_2^{l-1} f_1$ is obtained from f by application of the multiplier with the following symbol:

$$\frac{(2\pi i n)^{l-1}}{((2\pi i m)^k - \tau(2\pi i n)^l)}, \quad (m, n) \neq (0, 0).$$

Again, this follows from Lemma 1.10. See the hints to the proof of Lemma 2.2. □

The above lemma yields a statement in the spirit of Theorem 0.3 for the torus, yet in the L^1-metric.

Lemma 3.4. *Let proper trigonometric polynomials μ_j, $j = 0, \ldots, N$, and φ_j, $j = 1, \ldots, N$, satisfy* (0.3). *Then*

$$\sum_{j=1}^{N} \|\partial_1^{k-1} \varphi_j\|_{L_1(\mathbb{T}^2)}, \ \sum_{j=1}^{N} \|\partial_2^{l-1} \varphi_j\|_{L_1(\mathbb{T}^2)} \leqslant C \sum_{j=0}^{N} ||\mu_j||_{L^1(\mathbb{T}^2)},$$

where C does not depend on the functions involved.

Proof. We argue much as we did when deriving Theorem 0.4 from Theorem 3.1. Specifically, we construct the functions φ_{s_v} in the same way as in that argument. The numbers s_v must be chosen so that the $i^{l-k} s_v$ be pure imaginary. We apply Lemma 3.3 to show that the L^1-norms of the functions $\partial_1^{k-1}\varphi_{s_v}$ and $\partial_2^{l-1}\varphi_{s_v}$ are bounded in terms of the right-hand side of (0.4). But we saw that the initial

functions φ_j are linear combinations of the φ_{s_v} if we involve at least N mutually distinct parameters s_v (a system of linear equations with a Vandermonde determinant arises). So, we are done. $\square$

We need yet another fairly weak auxiliary estimate.

Lemma 3.5. *Under the assumptions of Lemma* 3.4, *we have*

$$\sum_{j=1}^{N} \|\varphi_j\|_{L^2(\mathbb{T}^2)} \leqslant C \sum_{j=0}^{N} ||\mu_j||_{L^1(\mathbb{T}^2)}.$$

Proof. If $k = l = 1$, this was established in [KM1]. Otherwise, we proceed as in the above proof. Specifically, we observe that, if the Fourier coefficients of two functions f_1 and f are related by formula (3.17) and f is integrable over the torus, then f_1 is square integrable with an appropriate norm estimate, because the quantities $((2\pi im)^k - \tau(2\pi in)^l)^{-1}$ form a square summable sequence whenever $k, l \in \mathbb{N}$ and $\max(k, l) > 1$. Next, we again invoke the functions φ_{s_v} as in the proof of Lemma 3.4, etc. $\square$

After these preparations, we can carry Theorem 0.4 over to the torus. Consider a cut-off function χ on $\mathbb{R}^2$. We choose it nonnegative, finitely supported, infinitely differentiable, and equal to 1 on $[-2, 2] \times [-2, 2]$. We define an operator P on smooth functions on the torus by the formula $P(f) = \tilde{f}\chi$, where $\tilde{f}$ is the periodic extension of f to the plane. Let a, b be nonnegative integers. Then P maps the space $W_2^{a,b}(\mathbb{T}^2)$ continuously to the space $\tilde{W}_2^{a,b}(\mathbb{R}^2)$ whose norm is defined by the formula $\|f\|_{\tilde{W}_2^{a,b}(\mathbb{R}^2)} = \left\|(1 + |\xi|)^a(1 + |\eta|)^b \hat{f}(\xi, \eta)\right\|_{L^2(\mathbb{R}^2)}$ (we have simply incorporated the junior derivatives in the Sobolev norm on the plane; recall that, for the torus, the junior derivatives were involved in the norm from the outset). Moreover, P is an isomorphism onto its image, i.e., $\|f\|_{W_2^{a,b}(\mathbb{T}^2)} \leqslant C\|Pf\|_{\tilde{W}_2^{a,b}(\mathbb{R}^2)}$, because the norms in the spaces involved are equivalent to the sum of the L^2-norms of the functions $\partial_1^u \partial_2^v f$ with $u \leqslant a$, $v \leqslant b$ and χ is identically 1 on the unit square.

We want to show that P possesses the same properties on Hilbert-type Sobolev spaces of nonintegral order. So, let a and b be nonnegative reals. The fact that P is still bounded from $W_2^{a,b}(\mathbb{T}^2)$ to $\tilde{W}_2^{a,b}(\mathbb{R}^2)$ is fairly easy. For example, we can argue by complex interpolation. Alternatively, we can observe that the Fourier transform of Pf is the sum of shifts of $\hat{\chi}$ times Fourier coefficients of f, and a direct estimate based on the rapid decay of $\hat{\chi}$ is possible. The proof of the fact that P is an isomorphism onto its image is more tricky. We introduce an operator Q from $\tilde{W}_2^{a,b}(\mathbb{R}^2)$ to periodic functions by putting

$$Qg(x, y) = \sum_{m,n \in \mathbb{Z}} g(x + m, y + n)\chi(x + m, y + n). \tag{3.18}$$

It can easily be seen that Q takes $\tilde{W}_2^{a,b}(\mathbb{R}^2)$ boundedly to $W_2^{a,b}(\mathbb{T}^2)$ if a and b are nonnegative integers. By interpolation, the same is true if a, b are nonnegative reals. Now, we look at the operator QP, which acts on $W_2^{a,b}(\mathbb{T}^2)$. It is easily seen that QP is multiplication by the function $w(x, y) = \sum_{m,n} \chi^2(x + m, y + n)$. This function is strictly positive and infinitely differentiable, hence QP has bounded inverse (this inverse is multiplication by $1/w$). It follows that P is also an isomorphism onto its image.

Now, we can finish the proof of Theorem 0.3. As in the proof of Theorem 0.4, it suffices to consider the case where one of the parameters α, β related by (0.5) is zero. Let, for definiteness, $\alpha = 0$; we prefer to retain the notation β for the second parameter in this case rather that to evaluate it. Recall that we assume all objects involved in (0.3) to be trigonometric polynomials. Consider the functions $P\varphi_j$ on the plane, then

$$\sum_{j=1}^N \|\varphi_j\|_{W_2^{0,\beta}(\mathbb{T}^2)} \leqslant C \sum_{j=1}^N \|P\varphi_j\|_{\tilde{W}_2^{0,\beta}(\mathbb{R}^2)} \leqslant C \sum_{j=1}^N \Big(\|P\varphi_j\|_{W_2^{0,\beta}(\mathbb{R}^2)} + \|P\varphi_j\|_{L^2(\mathbb{R}^2)} \Big).$$

The terms on the right that involve the L^2-norms of the functions $P\varphi_j$ themselves are estimated in terms of the $L^1(\mathbb{T}^2)$-norms of the μ_j with the help of Lemma 3.5. To treat the remaining terms, we introduce the following functions $f_0, \dots, f_N$ on the plane:

$$-\partial_1^k P\varphi_1 = f_0; \quad \partial_2^l P\varphi_j - \partial_1^k P\varphi_{j+1} = f_j, \quad j \in 1, \dots, N-1; \quad \partial_2^l P\varphi_N = f_N.$$

By Theorem 0.4, we have

$$\sum_{j=1}^N \|P\varphi_j\|_{W_2^{\alpha,\beta}(\mathbb{R}^2)} \leqslant C \sum_{j=0}^N \|f_j\|_{L^1(\mathbb{R}^2)}.$$

It remains to estimate the norms $\|f_j\|_{L^1(\mathbb{R}^2)}$ in terms of the quantities $\|\mu_j\|_{L^1(\mathbb{T}^2)}$. By construction, we have

$$f_j = \partial_2^l P\varphi_j - \partial_1^k P\varphi_{j+1} = \partial_2^l(\chi\tilde{\varphi}_j) - \partial_1^k(\chi\tilde{\varphi}_{j+1}) =$$
$$\chi\tilde{\mu_j} + \sum_{q=1}^l \binom{l}{q} \partial_2^q \chi \partial_2^{l-q} \tilde{\varphi}_j - \sum_{q=1}^k \binom{k}{q} \partial_1^q \chi \partial_1^{k-q} \tilde{\varphi}_{j+1}$$

for $j = 1, \dots, N-1$ (the formula differs slightly for $j = 0, N$). Consequently, we have the inequality

$$\|f_j\|_{L^1(\mathbb{R}^2)} \leqslant C\|\mu_j\|_{L^1(\mathbb{T}^2)} + C\|\varphi_j\|_{W_1^{0,l-1}(\mathbb{T}^2)} + C\|\varphi_j\|_{W_1^{k-1,0}(\mathbb{T}^2)}.$$

Finally, by 3.3, we obtain

$$\sum_{j=0}^N \|f_j\|_{L^1(\mathbb{R}^2)} \leqslant C \sum_{j=0}^N \|\mu_j\|_{L^1(\mathbb{T}^2)}.$$

This finishes the proof.

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